Rigid Rotations

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Abstract

We apply elementary Newtonian methods to rotating objects and develop the moment of inertia matrix. Then we proceed to the wobble of torque free gyroscope precession and other tumbling issues. The analysis of cases where angular momentum is not parallel to angular velocity is usually pursued through infinitesimal rotations. Here we work with more elementary ideas involving time derivatives of rotation matrices.

Introduction

We will discuss 3-d rotations about the origin on R^3 in terms of 3x3 matrices with the usual $\hat{x}, \hat{y}, \hat{z}$ basis. The set of matrices that correspond to rigid rotations are orthonormal and have determinant +1 (See Appendix B). This special rotation group of orthogonal matrices is denoted by SO_3 .

Basic Rotations

In consideration of rotations about the basis vectors, there is a natural differentiable parametrization of SO_3 from R^3 : $\vec{\theta} \to [s(\vec{\theta})]$. The square bracket notation emphasizes the matrix nature of the image of $\vec{\theta}$. The parametrization is defined by composing rotations about the three basic coordinate axes as:

$$[s(\vec{\theta})] = [s(\vec{\theta}_z)][s(\vec{\theta}_y)][s(\vec{\theta}_x)] \equiv \begin{pmatrix} a_z & -b_z & 0 \\ b_z & a_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_y & 0 & b_y \\ 0 & 1 & 0 \\ -b_y & 0 & a_y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_x & -b_x \\ 0 & b_x & a_x \end{pmatrix}$$

Where:

$a_x = \cos\theta_x \quad b_x = \sin\theta_x$ specifies rotation in the y-z plane

$a_y = \cos\theta_y \quad b_y = \sin\theta_y$ specifies rotation in the z-x plane

$a_z = \cos\theta_z \quad b_z = \sin\theta_z$ specifies rotation in the x-y plane

and $\vec{\theta} = (\theta_{1,} \theta_{2,} \theta_3) = \vec{\theta}_x + \vec{\theta}_y + \vec{\theta}_z$

Note: $[s(\vec{\theta}_1)][s(\vec{\theta}_2)] \neq [s(\vec{\theta}_1 + \vec{\theta}_2)]$ unlike 2-d rotations.

For small values of $\vec{\theta}$ the order of application of the $[s(\theta)_*]$'s , almost doesn't matter.

This parametrization covers all of SO_3 (See appendix D). But it is only one-to-one for a smaller rectangular solid region $-\pi < \theta_x < \pi$, $-\pi/2 < \theta_y < \pi/2$, $-\pi < \theta_z < \pi$.(See appendix E)

Paths of Rotations

The velocity and acceleration of a point exposed to continuous rotation can be determined from appropriate time derivatives. We express the continuous rotation as a path of rotations parametrized by time denoted by $\left[s\left(\vec{\theta}(t)\right)\right]$. We express the spacial path of an arbitrary point $\vec{r}_0$ exposed to these rotations as $\vec{r}(t)=\left[s\left(\vec{\theta}(t)\right)\right]\vec{r}_0$ The velocity of the point is the time derivative of its path $\dot{\vec{r}}(t)=\left(\frac{d}{dt}\left[s\left(\vec{\theta}(t)\right)\right]\right)\vec{r}_0$.

We simplify our analysis of $\frac{d}{dt}\left[s\left(\vec{\theta}(t_0)\right)\right]$ by restricting ourselves to the case where $\vec{\theta}(t_0)=\vec{0}$ and we have

$$\left[s\left(\vec{\theta}(t_0)\right)\right]=\left[s\left(\vec{0}\right)\right]=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=[1]$$

Then $\frac{d}{dt}\left[s\left(\vec{\theta}(t_0)\right)\right]=$

$$\begin{pmatrix}-\dot{\theta}_z b_z & -\dot{\theta}_z a_z & 0\\ \dot{\theta}_z a_z & -\dot{\theta}_z b_z & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a_y & 0 & -b_y\\ 0 & 1 & 0\\ b_y & 0 & a_y\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & a_x & -b_x\\ 0 & b_x & a_x\end{pmatrix}+$$

$$\begin{pmatrix}a_z & -b_z & 0\\ b_z & a_z & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\dot{\theta}_y b_y & 0 & \dot{\theta}_y a_y\\ 0 & 0 & 0\\ -\dot{\theta}_y a_y & 0 & -\dot{\theta} b_y\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & a_x & -b_x\\ 0 & b_x & a_x\end{pmatrix}+$$

$$\begin{pmatrix}a_z & -b_z & 0\\ b_z & a_z & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_y & 0 & -b_y\\ 0 & 1 & 0\\ b_y & 0 & a_y\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & -\dot{\theta}_x b_x & -\dot{\theta}_x a_x\\ 0 & \dot{\theta}_x a_x & -\dot{\theta}_x b_x\end{pmatrix}$$

When $t=t_0$, $\frac{d}{dt}\left[S\left(\left(\vec{\theta}(t_0)\right)\right)\right]=\frac{d}{dt}\left[S\left(\vec{0}\right)\right]=$

$$\begin{pmatrix} 0 & -\dot{\theta}_z & 0 \\ \dot{\theta}_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}+$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & \dot{\theta}_y \\ 0 & 0 & 0 \\ -\dot{\theta}_y & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}+$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_x \\ 0 & \dot{\theta}_x & 0 \end{pmatrix}=\begin{pmatrix} 0 & -\dot{\theta}_z & \dot{\theta}_y \\ \dot{\theta}_z & 0 & -\dot{\theta}_x \\ -\dot{\theta}_y & \dot{\theta}_x & 0 \end{pmatrix}$$

The application of the resulting matrix is equivalent to a cross product (See Appendix A). We can naturally superpose the parameter space which is isomorphic to R^3 on top of the original space R^3 with corresponding bases were $\hat{\theta}_x \to \hat{x}$, $\hat{\theta}_y \to \hat{y}$, and $\hat{\theta}_z \to \hat{z}$. In this case, we get:

$$\dot{\vec{r}}=\vec{v}=\frac{d}{dt}\left[s\left(\vec{\theta}(t_0)\right)\right]\vec{r}=\dot{\vec{\theta}}\times\vec{r}=\vec{\omega}\times\vec{r} \quad \text{at} \quad \vec{r}=\vec{r}_0 \quad .$$

When $\vec{\omega}$ is constant, we find that it points along the axis of rotation.

We have shown that $\frac{d}{dt}\left[s\left(\vec{\theta}(t_0)\right)\right]$ can be associated with a vector along the axis of rotation. However, there are problems with an analogous vectorial association with $\left[s\left(\vec{\theta}(t_0)\right)\right]$ (see Appendix F).

More strenuous matrix differentiation shows that:

$$\ddot{\vec{r}}=\vec{a}=\frac{d^2}{dt^2}\left[s\left(\vec{\theta}(t)\right)\right]\vec{r} = \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)+\vec{\alpha}\times\vec{r} \quad \text{at} \quad \vec{r}=\vec{r}_0 \quad .$$

where $\vec{\alpha}\equiv\left(\ddot{\theta}_x-\dot{\theta}_y\dot{\theta}_z \, , \, \ddot{\theta}_y+\dot{\theta}_x\dot{\theta}_z \, , \, \ddot{\theta}_z-\dot{\theta}_x\dot{\theta}_y\right)$ (See appendix G)

The two terms of $\ddot{\vec{r}}$ show centripetal and tangential plus other non-radial acceleration. The $\dot{\theta}_i\dot{\theta}_j$ terms occur when $\vec{\omega}$ moves away from a coordinate axis.

The derivative expressions away from from $\left[s(\vec{0})\right]=[1]$ are discouragingly complicated. Also for a constant angular velocity $\vec{\omega}$,the parametrized rotation given by $\left[s(t\vec{\omega})\right]$ does not represent uniform rotational motion unless $\vec{\omega}$ points along a coordinate axis. Differential analysis of a tumbling body is managed by moving the origin of the coordinate system when derivative expressions are required at some point.

Dynamics
Armed with expressions for acceleration, we can work with Newton's second law to determine rotational dynamics.

Let $\vec{F}$ be a force on a mass m at $\vec{r}$ subject to rotation about the origin.
Then $\vec{F}=m\ddot{\vec{r}}$, and

$$\vec{\tau}\equiv\vec{r}\times\vec{F}=m\,\vec{r}\times(\vec{\alpha}\times\vec{r})+m\,\vec{r}\times(\vec{\omega}\times(\vec{\omega}\times\vec{r}))$$

use cyclic identity $=-m\,\vec{r}\times(\vec{r}\times\vec{\alpha})-m\,\vec{\omega}\times((\vec{\omega}\times\vec{r})\times\vec{r})-m(\vec{\omega}\times\vec{r})\times(\vec{r}\times\vec{\omega})$

$$=-m\,\vec{r}\times(\vec{r}\times\vec{\alpha})-m\,\vec{\omega}\times(\vec{r}\times(\vec{r}\times\vec{\omega}))$$

Note that $\vec{r}\times(\vec{r}\times\vec{\alpha})=[\vec{r}\times]^2\vec{\alpha}=-\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}\vec{\alpha}$

(See appendix A)

Define $\mathrm{I}\equiv-m[\vec{r}\times]^2=m\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$ the moment of inertia.

and its associated integral form for continuous distributed mass,

$$\mathrm{I}\equiv-\int[\vec{r}\times]^2dm=\int\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}dm$$

We get $\vec{\tau}=\mathrm{I}\vec{\alpha}+\vec{\omega}\times\mathrm{I}\vec{\omega}$

which becomes $\vec{\tau}=\mathrm{I}\vec{\alpha}$ when $\vec{\omega}$ is an eigenvector of I

Moment of Inertia Tensor

The matrix $\mathrm{I}(\vec{r})$ represents a physical property of an object. Its behavior should not change when expressed in a rotated coordinate system T, in the sense that $\left(\mathrm{I}(T\vec{r})\right)(T\vec{\omega})=T\left(\mathrm{I}(\vec{r})\,\vec{\omega}\right)$ or equivalently $\mathrm{I}(T\vec{r})=T\,\mathrm{I}(\vec{r})\,T^{-1}$.

Mathematically this means that the definition of $\mathrm{I}(\vec{r})$ is linear enough. In this case, the definition of $\mathrm{I}(\vec{r})$ is founded on cross products which satisfy

$T\vec{u}\times T\vec{v}=T(\vec{u}\times\vec{v})$ for any rotation T.

In detail

$$\left(\mathrm{I}(T\vec{r})\right)(T\vec{\omega})=-m\left[(T\vec{r})\times\right]^2(T\vec{\omega})=-m\;T\vec{r}\times(T\vec{r}\times T\vec{\omega})$$
$$=-m\;T\left(\vec{r}\times(\vec{r}\times\vec{\omega})\right)=-m\;T[\vec{r}\times]^2\vec{\omega}=\left(T(-m)\,[\vec{r}\times]^2\right)\vec{\omega}=T\left(\mathrm{I}(\vec{r})\,\vec{\omega}\right)$$

Since I obeys the natural transformation law $\mathrm{I}(T\vec{r})=T\,\mathrm{I}(\vec{r})\,T^{-1}$ for rotations T, it is reasonable to extend the definition of I to all coordinate systems using the same transformation law, thus create I the tensor.

Energy and Momentum

We define angular momentum as: $\vec{L}\equiv\mathrm{I}\,\vec{\omega}$

then $\vec{\tau}=\mathrm{I}\,\vec{\alpha}+\vec{\omega}\times\vec{L}$

Kinetic energy for a point mass calculates as:

$$K=\frac{1}{2}m\,\vec{v}\cdot\vec{v}=\frac{1}{2}m\,(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})=-\frac{1}{2}m\left(\vec{\omega}^t[\times\vec{r}]\right)\cdot\left([\vec{r}\times]\vec{\omega}\right)$$
$$=-\frac{1}{2}m\,\vec{\omega}^t[\times\vec{r}][\vec{r}\times]\vec{\omega}=-\frac{1}{2}\vec{\omega}^t m[\vec{r}\times][\vec{r}\times]\vec{\omega}=\frac{1}{2}\vec{\omega}^t\,\mathrm{I}\,\vec{\omega}=\frac{1}{2}\vec{\omega}\cdot\vec{L}$$

The version using the integral of distributed mass works out the same.

Conservation of Angular Momentum

We have an expression involving torque and angular acceleration. The expression for torque and angular momentum is encouragingly simple. In spite of the possibility of the moment of inertia changing during complicated rotation.

$$\frac{d}{dt}\vec{L}=\frac{d}{dt}(\mathrm{I}\,\vec{\omega})=-m\;\frac{d}{dt}\left(\vec{r}\times(\vec{r}\times\vec{\omega})\right)=m\;\frac{d}{dt}\left(\vec{r}\times\dot{\vec{r}}\right)$$
$$=m\;\dot{\vec{r}}\times\dot{\vec{r}}+m\;\vec{r}\times\ddot{\vec{r}}=m\;\vec{r}\times\left(\vec{\omega}\times(\vec{\omega}\times\vec{r})+\vec{\alpha}\times\vec{r}\right)$$
$$=-m\;\vec{r}\times\left((\vec{\omega}\times\vec{r})\times\vec{\omega}\right)-m\;\vec{r}\times(\vec{r}\times\vec{\alpha})$$

$$=m\,(\vec{\omega}\times\vec{r})\times(\vec{\omega}\times\vec{r})+m\;\vec{\omega}\times(\vec{r}\times(\vec{\omega}\times\vec{r}))-m\;\vec{r}\times(\vec{r}\times\vec{\alpha})\quad \text{cyclic identity}$$

$$=-\vec{\omega}\times m\,(\vec{r}\times(\vec{r}\times\vec{\omega}))-m\;\vec{r}\times(\vec{r}\times\vec{\alpha})$$

$$=\vec{\omega}\times \mathrm{I}\,\vec{\omega}+\mathrm{I}\,\vec{\alpha}=\vec{\tau}$$

So, $\vec{\tau}=\frac{d}{dt}\vec{L}$ and, $\vec{L}$ is constant when $\vec{\tau}=\vec{0}$.

Conical Pendulum Example

The conical pendulum is the simplest case where the angular momentum is not parallel to the angular velocity. In the familiar case of a conical pendulum, torque is provided by gravity to continuously change the direction of the angular momentum. In this analysis, we describe the rotation of the system, calculate the changing angular momentum, and deduce the corresponding torque

Consider a mass swinging in a circle suspended by a rod.

For geometrical analysis, one end of the rod is fixed at the origin, and our coordinate system has $\hat{z}$ pointing down. The location of the mass is given by $\vec{r}=(x_t, y_t, z)$, where $x_t=r\,\sin\phi\,\cos\,t\omega_z$, $y_t=r\,\sin\phi\,\sin\,t\,\omega_z$ and $z=r\cos\phi=\text{const}$. In this case, we have $\vec{\theta}=(0,0,t\omega_z)$. Given this motion, we can deduce the torque (provided by gravity).

We calculate I at t=0:

$$\mathrm{I}=m\begin{pmatrix} y^2+z^2 & -x\,y & -x\,z \\ -x\,y & x^2+z^2 & -y\,z \\ -x\,z & -y\,z & x^2+y^2 \end{pmatrix}=m\,r^2\begin{pmatrix} \cos^2\phi & 0 & -\sin\phi\cos\phi \\ 0 & 1 & 0 \\ -\sin\phi\cos\phi & 0 & \sin^2\phi \end{pmatrix}$$

So, at t=0,

$$\vec{L}=m\,r^2\begin{pmatrix} \cos^2\phi & 0 & -\sin\phi\cos\phi \\ 0 & 1 & 0 \\ -\sin\phi\cos\phi & 0 & \sin^2\phi \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}=m\,r^2\sin\phi\begin{pmatrix} -\omega_z\cos\phi \\ 0 \\ \omega_z\sin\phi \end{pmatrix}$$

Since $\vec{\alpha}=\vec{0}$, We get

$$\vec{\tau}=\vec{\omega}\times\vec{L}=(0,0,\omega_z)\times m\,r^2\sin\phi\,(-\omega_z\cos\phi\,,0,\,\omega_z\sin\phi)$$

$$= -m r^2 \omega_z^2 \sin\phi \cos\phi \quad \hat{y}$$

This torque is directed toward the z-axis which we will show is related to the expected centripetal force.

Let $\vec{r}_\perp \equiv (x_t, y_t, 0)$,then $r_\perp = r \sin\phi$. The mass is traveling in a circle governed by a centripetal force given by : $\vec{F} = -m \frac{r_\perp^2 \omega_z^2}{r_\perp} \hat{x}$. Note that

$$\vec{r} \times \vec{F} = r \cos\phi \; \hat{z} \times F \hat{x} = r \cos\phi \; \hat{z} \times \left(-m r \sin\phi \, \omega_z^2 \; \hat{x}\right)$$

$$= -m r^2 \omega_z^2 \sin\phi \cos\phi \quad \hat{y} = \vec{\tau}$$

In this example $\vec{L}$ and $\vec{\omega}$ are not parallel. As time evolves, $\vec{L}$ moves around the z-axis with the rotation.

We can check kinetic energy by calculating:

$$K = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \omega_z \hat{z} \cdot m r^2 \sin\phi \left(-\omega_z \cos\phi, 0, \omega_z \sin\phi\right)$$

$$= \frac{1}{2} m r^2 \sin^2\phi \; \omega_z^2 = \frac{1}{2} m v^2$$

Principle Axes

The moment of inertia matrix $\mathtt{I}$ is positive, because $-[\vec{r}\times]^2$ is (see Appendix A). Furthermore, it is positive definite if there is some mass in all directions. This condition can be stated as:

For any $\vec{\omega}\neq\vec{0}$ there is some mass density at position $\vec{r}$ were $\vec{\omega}\times\vec{r}\neq\vec{0}$.

This rules out cases of rotating an infinitesimally thin rod about its axis.
So $\mathtt{I}$ has three positive eigenvalues $(\lambda_{1,}\lambda_{2,}\lambda_3)$ and three corresponding orthogonal unit eigenvectors. $(\hat{e}_1 , \hat{e}_2 , \hat{e}_3)$ (usually listed in order of increasing eigenvalue). The $\pm$ directions of the unit eigenvectors are chosen so that they form a right handed coordinate system and the previous sign conventions still apply. When $\mathtt{I}$ is calculated relative to the center of mass, the unit eigenvectors define the principle axes.

In the torque free case, we find that $\vec{\omega}$ does not change when it is an eigenvector. This follows from $\vec{0}=\vec{\tau}=\mathtt{I}\,\vec{\alpha}+\vec{\omega}\times\vec{L}$, so that:

$$\vec{\alpha}=-\mathtt{I}^{-1}(\vec{\omega}\times\vec{L})=-\mathtt{I}^{-1}(\vec{\omega}\times\mathtt{I}\,\vec{\omega})=0$$

However, we will see later that this rotation is stable only when $\vec{\omega}$ points in the direction of an extreme eigenvalue, $\hat{e}_1$ or $\hat{e}_3$

If the $(\hat{e}_1 , \hat{e}_2 , \hat{e}_3)$ coordinate system is used for $(\hat{x},\hat{y},\hat{z})$, $\mathtt{I}$ becomes a diagonal matrix giving an inequality related to the perpendicular axis theorem.

$$\lambda_3=\int (x^2+y^2)dm \leq \int (x^2+z^2)dm + \int (y^2+z^2)dm = \lambda_2+\lambda_1$$

Gyroscope Example

Consider a wheel and axle where the axle is initially aligned with $\hat{x}$ which is the principle axis with largest eigenvalue. The other two principle axes span the wheel and have equal eigenvalues. As before, we specify the rotational motion of the body and deduce the torque. We than get an expression for precessional frequency.

The axle of the wheel precesses in the xy-plane and has an associated angular velocity given by $\vec{\omega}_{xy}=\hat{x}\,\omega_{xy}\cos\omega_z t+\hat{y}\,\omega_{xy}\sin\omega_z t$ where $\vec{\omega}_z$ is a constant vector parallel to the z-axis and ω_{xy} is a constant scalar.

Evidently, the wheel assembly has a changing angular velocity given by $\vec{\omega}=\vec{\omega}_{xy}+\vec{\omega}_z$ with $\dot{\vec{\omega}}=-\hat{x}\,\omega_{xy}\omega_z\sin\omega_z t+\hat{y}\,\omega_{xy}\omega_z\cos\omega_z t$ orthogonal to the axle.

At t = 0 we have, $\vec{\alpha} \equiv (\dot{\omega}_x - \omega_y \omega_z \, , \, \dot{\omega}_y + \omega_x \omega_z \, , \, \dot{\omega}_z - \omega_x \omega_y) = (\omega_{xy}\omega_z + \omega_{xy}\omega_z)\hat{y}$

$\mathbb{I}$ has eigenvectors in the three coordinate directions, with last two of them equal.

$$\mathbb{I} = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{pmatrix} \text{ and } \vec{\tau} = \mathbb{I}\vec{\alpha} = 2\lambda_{yz}\omega_{xy}\omega_z \hat{y}$$

We find that $\omega_z = \dfrac{\tau}{2\lambda_{yz}\omega_{xy}}$

Momentum-Energy Ratios

Consider an eigenvector $\vec{\omega}_i$ of $\mathbb{I}$ at a specific time, then

$$L_i^2 = \vec{L}_i \cdot \vec{L}_i = (\mathbb{I}\omega_i) \cdot (\mathbb{I}\omega_i) = \lambda_i^2 \omega_i^2 \text{ and } K_i = \frac{1}{2}\vec{\omega}_i \cdot (\mathbb{I}\vec{\omega}_i) = \frac{1}{2}\lambda_i \omega_i^2$$

The ratio of kinetic energy and angular momentum expression about a principle axis must satisfy $\dfrac{L_i^2}{2K_i} = \lambda_i$ analogous to Newtonian particle dynamics where $\dfrac{p^2}{2K} = m$

Dumbbell Example

We have a mass m at either end of a rigid rod with length 2r. The analysis of its rotation about its center of mass is similar to that of a conical pendulum. This example shows the instantaneous change in angular speed when the continuously changing torque maintaining an oblique rotation is removed.

A constantly changing torque is used to force the dumbbell to rotate about its center with angular velocity $\vec{\omega}$, where $\vec{\omega}$ is at an angle ϕ from the rod. We get:

$$\vec{L} = 2mr^2\omega \sin\phi \; \hat{L}$$

where $\hat{L}$ is orthogonal to the rod and rotates with the system. Note that $\hat{L}$ always points in an eigenvector direction for this very symmetrical case.

Also $K = m r^2 \sin^2\phi \;\; \omega^2$, then $\frac{L^2}{2K} = 2 m r^2 = \lambda$

We find that the $L^2/2K$ ratio always leads to the only non-degenerate eigenvalue. In this idealized case $\lambda_1 = 0$ and $\lambda_2 = \lambda_3$. When the constraining force is released, the new $\vec{\omega}$ must point in the $\vec{L}$ direction because $\vec{L} = I\,\vec{\omega} = \lambda\,\vec{\omega}$. Also ω decreases because of the new axis of rotation and conservation of energy.

The angular momentum does not determine the kinetic energy, unlike the analogous notion in particle dynamics. But, the momentum-energy ratio is constrained.

Work-Energy Theorem

In Newtonian particle dynamics, one form of the work-energy theorem is $dK=\vec{F}\cdot d\vec{r}$. This can be developed from the power and momentum point of view as:

$$\frac{d}{dt}K=\frac{d}{dt}\left(\frac{1}{2}\vec{v}\cdot\vec{p}\right)=\frac{1}{2}\vec{a}\cdot\vec{p}+\frac{1}{2}\vec{a}\cdot m\vec{v}=\frac{1}{2}\vec{F}\cdot\vec{v}+\frac{1}{2}\vec{v}\cdot\vec{F}=\vec{F}\cdot\vec{v}$$ for any $\vec{v}$ along any path.

So $dK=\frac{d}{dt}K\,dt=\vec{F}\cdot\frac{d\vec{r}}{dt}dt=\vec{F}\cdot d\vec{r}$ along any path.

We will use an analogous approach for the rotational work-energy theorem. Note the regrouping of $\vec{a}\cdot\vec{p}$ to $m\vec{a}\cdot\vec{v}$. The analogous regrouping for rotational dynamics doesn't obviously work. However, when working with eigenvectors of $\mathtt{I}$, the eigenvalues λ_i can take the role of m.

Without loss of physical generality consider any path of rotations $\left[s\left(\vec{\theta}(t)\right)\right]$ where $\left[s\left(\vec{\theta}(0)\right)\right]=[1]$ and the coordinate axes are aligned with the eigenvectors.

We will work with the $i'th$ component $\vec{\omega}_i$ of angular velocity $\vec{\omega}$ along a path. This amounts to focusing on 2-d rotations about principle axes.

We have $\frac{d}{dt}K=\sum_i\frac{d}{dt}K_i$ where $K_i=\frac{1}{2}\vec{\omega}_i\cdot\lambda_i\vec{\omega}_i=\frac{1}{2}\vec{\omega}_i\cdot\vec{L}_i$

We will encounter some issues involving the distinction between α and $\dot{\vec{\omega}}$,

To this end consider the three component paths defined by the basic angles:

$$\left[s_i(t)\right]\equiv\left[s\left(\vec{\theta}_i(t)\right)\right]$$

The corresponding component paths $\left[s_i(t)\right]$ have the same contributors to angular momentum $\vec{L}_i$ and energy K_i as the original path $\left[s\left(\vec{\theta}(t)\right)\right]$, at t=0, because these values only depend on $\vec{\omega}_i$

For each component path, the direction of the rotation vector isn't changing, so that $\dot{\omega}_i = \alpha_i$.

Calculation with a component path gives:

$$\frac{d}{dt}K_i = \frac{d}{dt}\left(\frac{1}{2}\vec{\omega}_{ia}\cdot\vec{L}_i\right) = \frac{1}{2}\vec{\alpha}_i\cdot\vec{L}_i + \frac{1}{2}\vec{\omega}_i\cdot\vec{\tau}_i = \frac{1}{2}\vec{\alpha}_i\cdot\lambda\,\vec{\omega}_i + \frac{1}{2}\vec{\omega}_i\cdot\vec{\tau}_i$$

$$= \frac{1}{2}\lambda\,\dot{\vec{\omega}}_i\cdot\vec{\omega}_i + \frac{1}{2}\vec{\omega}_i\cdot\vec{\tau}_i = \frac{1}{2}\left(\mathrm{I}\,\vec{\alpha}\right)_i\cdot\vec{\omega}_i + \frac{1}{2}\vec{\omega}_i\cdot\vec{\tau}_i$$

$$= \frac{1}{2}\vec{\tau}_i\cdot\vec{\omega}_i + \frac{1}{2}\vec{\omega}_i\cdot\vec{\tau}_i = \vec{\tau}_i\cdot\vec{\omega}_i$$

So $dK_i = \frac{d}{dt}K_i\,dt = \vec{\tau}_i\cdot\frac{d\vec{\theta}_i}{dt}dt = \vec{\tau}_i\cdot d\vec{\theta}_i$ and

$dK = \frac{d}{dt}K\,dt = \vec{\tau}\cdot\frac{d\vec{\theta}}{dt}dt = \vec{\tau}\cdot d\vec{\theta}$ along the path $\left[s\left(\vec{\theta}(t)\right)\right]$ at t=0.

Momentum-Energy Ratio constraints

For a point mass, the kinetic energy can have only one value for a given momentum. The analogous equality doesn't exist for a rotating irregular body, however, there are constraints on the ratio $L^2/2K$.

The three principle directions will have three different eigenvalues still satisfying: $\frac{L_i^2}{2K_i} = \lambda_i$. We will show that the eigenvalues λ_1 and λ_3 are the extreme values on $L^2/2K$. These limits are related to the stability of various modes of rotation.

When rotating about a general axis, we can express the rotational velocity as the sum of orthogonal eigenvectors $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3$ (parallel to the principle axes).

Then $\vec{L} = \mathrm{I}\,\omega = \lambda_1\vec{\omega}_1 + \lambda_2\vec{\omega}_2 + \lambda_3\vec{\omega}_3 = \vec{L}_1 + \vec{L}_2 + \vec{L}_3$

and $L^2 = L_1^2 + L_2^2 + L_3^2 = \lambda_1^2\omega_1^2 + \lambda_2^2\omega_2^2 + \lambda_3^2\omega_3^2$

where the eigenvalues are ordered by increasing magnitude.

Also $K=\frac{1}{2}\vec{\omega}\cdot\vec{L}=\frac{1}{2}\left(\lambda_1\omega_1^2+\lambda_2\omega_2^2+\lambda_3\omega_3^2\right)=K_1+K_2+K_3$

So,

$$\frac{L^2}{2K}=\frac{\lambda_1^2\omega_1^2+\lambda_2^2\omega_2^2+\lambda_3^2\omega_3^2}{\lambda_1\omega_1^2+\lambda_2\omega_2^2+\lambda_3\omega_3^2}=\lambda_1\left(\frac{1+\left(\lambda_2^2/\lambda_1^2\right)\left(\omega_2^2/\omega_1^2\right)+\left(\lambda_3^2/\lambda_1^2\right)\left(\omega_3^2/\omega_1^2\right)}{1+\left(\lambda_2/\lambda_1\right)\left(\omega_2^2/\omega_1^2\right)+\left(\lambda_3/\lambda_1\right)\left(\omega_3^2/\omega_1^2\right)}\right)\geq\lambda_1$$

and

$$\frac{L^2}{2K}=\lambda_3\left(\frac{\left(\lambda_1^2/\lambda_3^2\right)\left(\omega_1^2/\omega_3^2\right)+\left(\lambda_2^2/\lambda_3^2\right)\left(\omega_2^2/\omega_3^2\right)+1}{\left(\lambda_1/\lambda_3\right)\left(\omega_1^2/\omega_3^2\right)+\left(\lambda_2/\lambda_3\right)\left(\omega_2^2/\omega_3^2\right)+1}\right)\leq\lambda_3$$

We have shown that $\lambda_1\leq\frac{L^2}{2K}\leq\lambda_3$ .by observing that the long fractions above are more or less than one according to case.

The solution values of $\vec{\omega}$ at extreme eigenvalues are unique (except for $\pm$). But there is a continuum of solutions when $\lambda_1<\frac{L^2}{2K}<\lambda_3$.

Torque Free Solution sets for $\vec{\omega}$

When no torque is applied to a tumbling irregular body, the angular momentum and energy doesn't change, but the angular velocity can vary. The magnitudes of energy and momentum put constraints on the possible values of the angular velocity.

For a given K and $\vec{L}$, the possible values for $\vec{\omega}$ must lie on the concentric ellipsoids $L^2=\lambda_1^2\omega_1^2+\lambda_2^2\omega_2^2+\lambda_3^2\omega_3^2$ and $2K=\lambda_1\omega_1^2+\lambda_2\omega_2^2+\lambda_3\omega_3^2$.

So, the possible $\vec{\omega}$'s lie on a pair of closed loops defined by the intersection of the two (Poinsot's) ellipsoids.

To get a sense of these curves, we will algebraically project onto the $\vec{\omega}_{1,}\vec{\omega}_3$ -plane by eliminating ω_2 .

From angular momentum, we have: $\lambda_2 \omega_2^2 = \frac{1}{\lambda_2}\left(L^2 - \lambda_1^2 \omega_1^2 - \lambda_3^2 \omega_3^2\right)$ which we substitute into the energy conservation equation and get:

$$\left(\lambda_1 - \lambda_1^2/\lambda_2\right)\omega_1^2 + \left(\lambda_3 - \lambda_3^2/\lambda_2\right)\omega_3^2 = 2K - L^2/\lambda_2$$

and rearrange as

$$\lambda_3\left(\lambda_3 - \lambda_2\right)\omega_3^2 - \lambda_1\left(\lambda_2 - \lambda_1\right)\omega_1^2 = L^2 - 2K\lambda_2 \quad \text{a hyperbola.}$$

When rotation is about an extreme axis, the hyperbola shrinks into its truncation points at the vertices of the ellipses defined by $2K$ and L^2 .

When rotation is about the axis with eigenvalue λ_2 , the hyperbola degenerates into two intersecting lines. In this case $L^2 / 2K = \lambda_2$.

When the object is symmetric about one of its axes (for example $\lambda_1 = \lambda_2$), the hyperbola degenerates into two parallel lines.

Stability of Rotation about Extreme Axes

For an an angular velocity which is initially close to an extreme principle axis, it will remain close to the same axis, This is because of the hyperbolic constraint on $\vec{\omega}$ near an extreme direction.

For an algebraic example , consider the case where $\vec{\omega}$ is initially almost parallel to $\hat{e}_3$ in the sense that:

$$\frac{\omega_1^2}{\omega_3^2} < \epsilon \quad \text{and} \quad \frac{\omega_2^2}{\omega_3^2} < \epsilon \quad \text{and} \quad \frac{\omega_1^2}{\omega_3^2} \text{ is small enough that } 0 \le L^2 - 2K\lambda_2$$

We can show algebraically that:

If $\lambda_1 \frac{\omega_1^2}{\omega_3^2} + \lambda_2 \frac{\omega_2^2}{\omega_3^2} < \epsilon_L$ initially,

then any subsequent angular velocity $\vec{\omega}_s$ satisfies

$$\lambda_1 \frac{\omega_{s1}^2}{\omega_{s3}^2} + \lambda_2 \frac{\omega_{s1}^2}{\omega_{s3}^2} \le C\ \epsilon_L$$

where $C = \frac{2K}{\lambda_3^2 \left(L^2 - 2K\lambda_2\right)}$, a constant (See Appendix H) .

So $\vec{\omega}_s$ remains almost parallel to $\hat{e}_3$.

Example: Torque Free Precession (nutation)

Having shown that $\vec{\omega}$ is constrained to a pair of curves, we show that $\vec{\omega}$ actually moves along the constraint curve by considering $\vec{\alpha}$.

In this example, $\lambda_1 = \lambda_2 < \lambda_3$ .and we think of a wheel with its axis along $\hat{e}_3$.

We will show that a point on the wheel axis undergoes uniform circular motion around $\vec{L}$.

We consider the wheel at any instant in time and find the coordinate system defined by the eigenvectors $\left(\hat{e}_{1,} \hat{e}_{2,} \hat{e}_3\right)$

Because $\lambda_1 = \lambda_2$ all vectors orthogonal to $\hat{e}_3$ are eigenvectors, we are free to choose $\hat{e}_1$ and $\hat{e}_2$ as $\hat{e}_1 = \hat{L} \times \hat{e}_3$ and $\hat{e}_2 = \hat{e}_3 \times \hat{e}_1$ with $\omega_1 = 0$.

Then $\vec{L} = \left(0, \lambda_2 \omega_2, \lambda_3 \omega_3\right)$

Let $\vec{r}_3 \equiv \hat{e}_3$ be the unit vector on the wheel axis. Then $\dot{\vec{r}}_3 = \left(\omega_2, 0, 0\right)$.

After determining $\vec{\alpha}$ we can find $\ddot{\vec{r}}_3$ as

$$\ddot{\vec{r}}_3 = \omega_2 \left(0\ ,\ \frac{\lambda_3}{\lambda_1} \omega_3\ ,\ -\omega_2\right)$$ (See Appendix I).

So $\dot{\vec{r}}_3$, $\ddot{\vec{r}}_3$ and $\vec{L}$ are mutually orthogonal with $\ddot{\vec{r}}_3$ pointing toward $\vec{L}$ because $\ddot{\vec{r}}_3 \cdot \left(\vec{L} - \vec{r}_3\right) = \omega_2^2 > 0$.

If $\vec{r}_3$ is revolving around $\vec{L}$ its radius ρ must be given by

$$\rho^2 = \left\|\vec{r}_3\right\|^2 - \frac{\vec{L}\cdot\vec{r}_3}{L^2} = 1 - \frac{L_3^2}{L^2} = \frac{\lambda_1^2\omega_2^2}{\lambda_2^2\omega_2^2 + \lambda_3^2\omega_3^2}$$

To show that ρ , $\left\|\dot{\vec{r}}_3\right\|$ and $\left\|\ddot{\vec{r}}_3\right\|$ have the correct relationship for uniform circular motion we calculate

$$\rho^2\left\|\ddot{\vec{r}}_3\right\|^2 = \frac{\lambda_1^2\omega_2^2}{\lambda_1^2\omega_2^2 + \lambda_3^2\omega_3^2}\;\frac{\omega_2^2\left(\lambda_3^2\omega_3^2 + \lambda_1^2\omega_2^2\right)}{\lambda_1^2} = \left\|\dot{\vec{r}}_3\right\|^4$$

So, we have uniform circular motion and the angular precession is given by:

$$\omega_p^2 = \frac{\left\|\dot{\vec{r}}_3\right\|^2}{\rho^2} = \omega_2^2\frac{L^2}{L^2 - L_3^2} = \frac{L^2}{\lambda_1^2}$$ which only changes with energy.

For large L, it gets harder to produce the initial conditions to demonstrate this motion.

Symmetry allows us to use the above calculations at any point in its motion.

Observe that this torque free motion can be superposed on any motion that includes torque. This motion superposed on a gyroscope experiencing gravitational torque is usually called nutation.

Example: Degenerate Rotation of a Sphere

When all three eigenvalues are the same, any vector is an eigenvector and the constraints given by L^2 and 2K do not explain the stable rotation of a symmetric object. However, if we examine the time derivative of the unit vector on the axis of rotation $\vec{r}_\omega = \hat{\omega}$, we find that $\dot{\vec{r}}_\omega = \vec{0}$ for any choice of $\vec{\omega}$ (See Appendix I).

Conclusion

We have described rigid rotation effects in terms of the direct dynamics of individual points of a rotating body. There are more general ways of treating the same material with Hamiltonian or Lagrangian techniques.

Appendix A: Cross Products

The cross product of two vectors (or vector product) results in a new vector perpendicular to the plane defined by the original two vectors (using the right hand rule).

The magnitude is given by $\|\vec{r}\times\vec{v}\|=\|\vec{r}\|\|\vec{v}\|\sin\theta$ and is the area of the parallelogram spanned by $\vec{r}$ and $\vec{v}$.

In Cartesian coordinates

$$\vec{r}\times\vec{v}=(r_2 v_3-r_3 v_2\ ,\ r_3 v_1-r_1 v_3\ ,\ r_1 v_2-r_2 v_1)$$

Cyclic Identity (usually used to manipulate nested cross products).

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) = 0$$

Matrix Version (useful for mixed matrix and vector expressions)

$$\vec{r}\times\vec{v} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \vec{v} \equiv [\vec{r}\times]\,\vec{v} \quad \text{and} \quad \vec{v}\times\vec{r} = \vec{v}^t \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \equiv \vec{v}^t[\times\vec{r}]$$

note that $[\vec{r}\times]=[\times\vec{r}]$ is used differently in left and right multiplication

also $$\vec{r}\times(\vec{r}\times\vec{v})=[\vec{r}\times]^2\,\vec{v} = \begin{pmatrix} -y^2-z^2 & xy & xz \\ xy & -x^2-z^2 & yz \\ xz & yz & -x^2-y^2 \end{pmatrix} \vec{v}$$

Furthermore $-[\vec{r}\times]^2$ is positive because

$$-\vec{v}^t[\vec{r}\times]^2\vec{v} = -(\vec{v}\times\vec{r})\cdot(\vec{r}\times\vec{v}) = (\vec{v}\times\vec{r})^2 \geq 0$$

Appendix B: Isometries and 3x3 matrices

We think of rigid rotations as transformations that keep one point fixed and relative distances don't change. There is the further restriction that a rotated axis system doesn't change its relative orientation. This is expressed as preservation of the right hand rule for a system of axes.

It is clear that every $[s] \in SO_3$ is a rigid rotation. We will show that every rotation can be expressed as an element of SO_3 ,

Consider a rigid rotation as a map $T: R^3 \to R^3$ that preserves distances and the right-hand rule. Then angles are preserved because transformed triangles are congruent. This means dot products are preserved because:

$$T(\vec{u}) \cdot T(\vec{v}) = \|T(\vec{u})\| \|T(\vec{v})\| \cos\theta = \|\vec{u}\| \|\vec{v}\| \cos\theta = \vec{u} \cdot \vec{v} \quad .$$

Similarly cross products are preserved, in the sense that $T(\vec{u} \times \vec{v}) = T(\vec{u}) \times T(\vec{v})$.

To show that T is linear, consider the two bases $\hat{e}_i$ and $T(\hat{e}_i)$ Then use the bi-linearity of dot products.

$$T(\alpha\vec{u} + \vec{v}) \cdot T(\hat{e}_i) = (\alpha\vec{u} + \vec{v}) \cdot \hat{e}_i = \alpha\vec{u} \cdot \hat{e}_i + \vec{v} \cdot \hat{e}_i = \alpha T(\vec{u}) \cdot T(\hat{e}_i) + T(\vec{v}) \cdot T(\hat{e}_i)$$

So, $T(\alpha\vec{u} + \vec{v}) = \alpha T(\vec{u}) + T(\vec{v})$, showing that T is linear and can be represented as a 3x3 matrix.

Now the 3x3 matrix that represents T must be orthonormal with determinant +1 because of preservation of cross product magnitudes and the right-hand rule.

So, $T \in SO_3$

Appendix C: Euler's Rotation Theorem

Any non-trivial element $[s] \in SO_3$ has an axis of rotation essentially because there is at least one real positive eigenvalue.

Note that the characteristic polynomial of $[s]$ is a cubic polynomial given by:

$$P(\lambda) = |[s] - [1]\lambda|$$

We know that it has one real root, but it may be negative. However, the leading coefficient is negative so that $P(\lambda) \to -\infty$ as $\lambda \to \infty$. Also $P(0) = +1$. So, there is a positive root.

The presence of an axis of rotation is independent of alternative orthonormal bases as long as the right hand rule doesn't change. So, we can choose an orthonormal (with determinant +1) transformation T to transform coordinates so that , $T^{-1}[s]T=\begin{pmatrix}\lambda_1 & c_{12} & c_{13}\\ 0 & c_{22} & c_{23}\\ 0 & c_{32} & c_{33}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & c_{22} & c_{23}\\ 0 & c_{32} & c_{33}\end{pmatrix}$

where we used the orthonormality of the matrices and $\lambda_1>0$

The sub-matrix $\begin{pmatrix}c_{22} & c_{23}\\ c_{32} & c_{33}\end{pmatrix}$ is orthonormal with determinant +1.

So, $T^{-1}[s]T$ represents a rotation about the x-axis. Also $T\hat{x}$ is the axis of rotation of $[s]$ because it remains stationary under the rotation $[s]$ as shown by the calculation:

$[s](T\hat{x})=T\left(T^{-1}[s]T\right)\hat{x}=T\hat{x}$

Appendix D: The Standard Parametrization is Onto

Although it seems obvious that every $[s]\in SO_3$ is in the range of $[s(\vec{\theta})]$, it takes some busy work to show.

Let $[s]=\begin{pmatrix}s_{11} & s_{12} & s_{13}\\ s_{21} & s_{22} & s_{23}\\ s_{31} & s_{32} & s_{33}\end{pmatrix}\in SO_3$

We will invert $[s]$ by finding a sequence of basic rotations and get

$$[c_x][c_y][c_z][s]=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Inverting this composition, gives the required elementary rotations.

if $s_{11}^2+s_{21}^2\neq 0$ let

$$a_{21}=\frac{s_{11}}{\sqrt{s_{11}^2+s_{21}^2}}\qquad b_{21}=\frac{s_{21}}{\sqrt{s_{11}^2+s_{21}^2}}\qquad [c_z]=\begin{pmatrix}a_{21} & b_{21} & 0\\ -b_{21} & a_{21} & 0\\ 0 & 0 & 1\end{pmatrix}$$

Then $[c_z][s]=\begin{pmatrix}s_{11z} & s_{12z} & s_{13z}\\ 0 & s_{22z} & s_{23z}\\ s_{31} & s_{32} & s_{33}\end{pmatrix}$ with $s_{11z}>0$ and $s_{11}^2+s_{31z}^2=1$

To determine $[c_y]$ let

$$a_{31}=s_{11z} \quad b_{31}=s_{31} \quad [c_y]=\begin{pmatrix} a_{31} & 0 & b_{31} \\ 0 & 1 & 0 \\ -b_{31} & 0 & a_{31} \end{pmatrix}$$

Then $[c_y][c_z][s]=\begin{pmatrix} s_{11y} & s_{12y} & s_{13y} \\ 0 & s_{22z} & s_{23z} \\ 0 & s_{32y} & s_{33y} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & s_{22z} & s_{23z} \\ 0 & s_{32y} & s_{33y} \end{pmatrix}$

because of orthonormality and $s_{11y}>0$

if $s_{11}^2+s_{21}^2=0$ then $s_{31}^2=1$.

let $[c_z]=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $[c_y]=\begin{pmatrix} 0 & 0 & s_{31} \\ 0 & 1 & 0 \\ -s_{31} & 0 & 0 \end{pmatrix}$

then

$[c_y][c_z][s]=\begin{pmatrix} 1 & s_{12y} & s_{13y} \\ s_{21z} & s_{22z} & s_{23z} \\ s_{31y} & s_{32y} & s_{33y} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & s_{22z} & s_{23z} \\ 0 & s_{32y} & s_{33y} \end{pmatrix}$ because of orthonormality

We have $s_{22z}^2+s_{32y}^2=1$ let

$$a_{32}=s_{22z} \quad b_{32}=s_{32y} \quad [c_x]=\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{32} & b_{32} \\ 0 & -b_{23} & a_{32} \end{pmatrix}$$

Then $[c_x][c_y][c_z][s]=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s_{23x} \\ 0 & 0 & s_{33x} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

because of orthonormality and determinant +1

So we get $[s]=[c_z]^{-1}[c_y]^{-1}[c_x]^{-1}$

The the angles for the basic rotations can be determined from the a_*'s and b_*'s which represent cos's and sin's. Inverting these rotations is accomplished by negating the angles.

The axis of rotation of $\left[s(\vec{\theta})\right]$ is usually not given by $\vec{\theta}$.

Appendix E: The Standard Parametrization is Locally One-to-One

$$\left[s(\vec{\theta})\right]=\begin{pmatrix} a_z & -b_z & 0 \\ b_z & a_z & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_y & 0 & b_y \\ 0 & 1 & 0 \\ -b_y & 0 & a_y \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_x & -b_x \\ 0 & b_x & a_x \end{pmatrix}$$

$$=\begin{pmatrix} a_y a_z & a_z b_x b_y - a_x b_z & a_x a_z b_y + b_x b_z \\ a_y b_z & b_x b_y b_z + a_x a_z & a_x b_y b_z - a_z b_x \\ -b_y & a_y b_x & a_x a_y \end{pmatrix}$$

Counterexample

Note that the parametrization is not one-to-one for a line of $\vec{\theta}$'s where $\theta_y=\pi/2$. Consider $\vec{\theta}=(\phi\ ,\ \pi/2\ ,\ \phi)$ then $b_y=1$, $a_y=0$ and

$$\left[s(\vec{\theta})\right]=\begin{pmatrix} 0 & a_z b_x - a_x b_z & a_x a_z + b_x b_z \\ 0 & b_x b_z + a_x a_z & a_x b_z - a_z b_x \\ -1 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & -\sin(\phi-\phi) & 1 \\ 0 & 1 & \sin(\phi-\phi) \\ -1 & 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ for any } \phi$$

This example is related to gimbal lock.

We note further that $\left[s(\theta_x-\pi\ ,\ \pi-\theta_y\ ,\ \theta_z-\pi)\right]=\left[s(\theta_x,\theta_y,\theta_z)\right]$

The one-to-one region

Consider the region $-\pi<\theta_x<\pi$, $-\pi/2<\theta_y<\pi/2$, $-\pi<\theta_z<\pi$ containing $\vec{\theta}_1$ and $\vec{\theta}_2$. Suppose $\left[s(\vec{\theta}_1)\right]=\left[s(\vec{\theta}_2)\right]$, and we consider the first column and last row of $[s(\vec{\theta})]$.

Then, we have $b_{1y}=b_{2y}$ and we get $\theta_{1y}=\theta_{2y}$, because we are restricted to a region where $b_y=\sin\theta_y$ is one-to-one.

Also, we are in a region where $a_y=\cos\theta_y\neq 0$ and we deduce that

$$a_{1z}=a_{2z} \;,\; b_{1z}=b_{2z} \;,\; b_{1x}=b_{2x} \;,\; a_{1x}=a_{2x} \;.$$

So we have $\theta_{1x}=\theta_{2x}$ and $\theta_{1z}=\theta_{2z}$ because the $\pm 2\pi n$ terms are disallowed.

Appendix F: Rotation Vectors

It is sometimes convenient to express rotational quantities as vectors along the axis of rotation. This can lead to mathematical inconsistency.

For $\vec{r}(t)=[s(\vec{\theta}(t))]\vec{r}_0$ we have shown that $\dot{\vec{r}}=\dot{\vec{\theta}}\times\vec{r}_0$

However, an analogous expression for displacement $\vec{r}=\vec{\theta}\times\vec{r}_0$ is only approximately valid for small $\vec{\theta}$.

And the analogous expression for tangential acceleration $\ddot{\vec{r}}=\ddot{\vec{\theta}}\times\vec{r}_0$ is only valid when $\ddot{\vec{\theta}}$ is parallel to one of the coordinate axes.

In this presentation, we distinguish between:

the parameter space where $\vec{\theta}\in R^3$ a vector space

the space of rotations where $[s]\in SO_3$ a set of matrices

the physical space where $\vec{r}\in R^3$ a vector space

Apparently $\dot{\vec{\theta}}$ is the only quantity that passes from the parameter space, through the rotation space into physical space without trouble.

It is already clear that although $\vec{\theta}$ is a vector in the parameter space, $[s(\vec{\theta})]$ is not a vector because of the non-linear map. Although $\dot{\vec{\theta}}$ is a vector in the parameter space, we need a reason why it is a vector in the physical space.

Because of the 3-d parametrization, we can view SO_3 as a 3-d manifold embedded in the 9-d space of 3x3 matrices. In this view, $\frac{d}{dt}\left[s\left(\vec{\theta}\left(t_0\right)\right)\right]$ is a vector in the 3-d tangent plane of SO_3 . When $\vec{\theta}\left(t_0\right)=[1]$, the parameter space can be identified with the tangent plane.

So, we can use relaxed notation and associate $\frac{d}{dt}\left[s\left(\vec{\theta}\left(t_0\right)\right)\right]$ and $\dot{\vec{\theta}}\left(t_0\right)$

When we superpose the parameter space over the physical space with aligned axes, It would be more precise to say that we are superposing the tangent space of SO_3 . So we find that $\dot{\vec{\theta}}\left(t_0\right)$ can be treated like a vector in the physical space.

We get an approximate similar result for $\vec{\theta}\approx\vec{0}$ where the tangent plane is close to SO_3 . We do not get a simple result for $\ddot{\vec{\theta}}\left(t_0\right)$ because of the curvature of SO_3 .

Incidentally, we have shown that the tangent spaces to SO_3 are the 3x3 skew symmetric matrices.

Appendix G: Second Derivative of a Path

We are calculating a second derivative of a path at $t = t_0$ where $\vec{\theta}(t_0) = \vec{0}$.

The six terms of the second derivative of the product of the three basic rotations can be expressed as:

$$\frac{d^2}{dt^2}\left[s\left(\vec{\theta}(t_0)\right)\right] = \ddot{s}_z s_y s_x + s_z \ddot{s}_y s_x + s_z s_y \ddot{s}_x + 2 s_z \dot{s}_y \dot{s}_x + 2 \dot{s}_z s_y \dot{s}_x + 2 \dot{s}_z \dot{s}_y s_x$$

Expanding by chain rules give:

$$\dot{s}_* = \dot{\theta}_* \frac{d}{d\theta_*} s_* \qquad \ddot{s}_* = \ddot{\theta}_* \frac{d}{d\theta_*} s_* + \dot{\theta}_*^2 \frac{d^2}{d\theta_*^2} s_* = \ddot{\theta}_* \frac{d}{d\theta_*} s_* - \dot{\theta}_*^2 \tilde{s}_*$$

Evaluating the derivatives at $\vec{\theta}(t_0) = \vec{0}$ gives:

$$\left.\frac{d}{d\theta_x} s_x\right|_0 = \left.\begin{pmatrix} 0 & 0 & 0 \\ 0 & -b_x & -a_x \\ 0 & a_x & -b_x \end{pmatrix}\right|_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \left.\frac{d}{d\theta_*} s_*\right|_0 = \cdots$$

$$\left.\dot{s}_x\right|_0 = \left.\dot{\theta}_x \frac{d}{d\theta_x} s_x\right|_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_x \\ 0 & \dot{\theta}_x & 0 \end{pmatrix} \qquad \left.\dot{s}_*\right|_0 = \cdots$$

$$\left.\ddot{\theta}_x \frac{d}{d\theta_x} s_x\right|_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\ddot{\theta}_x \\ 0 & \ddot{\theta}_x & 0 \end{pmatrix} \qquad \left.\ddot{\theta}_* \frac{d}{d\theta_*} s_*\right|_0 = \cdots$$

$$\left.\tilde{s}_x\right|_0 \equiv \left.\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_x & -b_x \\ 0 & b_x & a_x \end{pmatrix}\right|_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \left.\tilde{s}_*\right|_0 \equiv \cdots$$

$$\left.\dot{\theta}_x^2 \tilde{s}_x\right|_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dot{\theta}_x^2 & 0 \\ 0 & 0 & \dot{\theta}_x^2 \end{pmatrix} \qquad \left.\dot{\theta}_*^2 \tilde{s}_*\right|_0 = \cdots$$

Summing the matrix expressions give:

$$\sum \ddot{\theta}_* \frac{d}{d\theta_*} s_* \bigg|_0 = \begin{pmatrix} 0 & -\ddot{\theta}_z & \ddot{\theta}_y \\ \ddot{\theta}_z & 0 & -\ddot{\theta}_x \\ -\ddot{\theta}_y & \ddot{\theta}_x & 0 \end{pmatrix} = \left[\ddot{\vec{\theta}}\times\right]$$

$$\sum \dot{\theta}_*^2 \tilde{s}_*\Big|_0 = \begin{pmatrix} \dot{\theta}_y^2+\dot{\theta}_z^2 & 0 & 0 \\ 0 & \dot{\theta}_x^2+\dot{\theta}_z^2 & 0 \\ 0 & 0 & \dot{\theta}_x^2+\dot{\theta}_y^2 \end{pmatrix}$$

The $\dot{s}_i \dot{s}_j \Big|_0$ terms require more attention

$$\dot{s}_y \dot{s}_x \Big|_0 = \begin{pmatrix} 0 & 0 & \dot{\theta}_y \\ 0 & 0 & 0 \\ -\dot{\theta}_y & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_x \\ 0 & \dot{\theta}_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dot{\theta}_x \dot{\theta}_y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dot{s}_z \dot{s}_x \Big|_0 = \begin{pmatrix} 0 & -\dot{\theta}_z & 0 \\ \dot{\theta}_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_x \\ 0 & \dot{\theta}_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dot{\theta}_x \dot{\theta}_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dot{s}_z \dot{s}_y \Big|_0 = \begin{pmatrix} 0 & -\dot{\theta}_z & 0 \\ \dot{\theta}_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \dot{\theta}_y \\ 0 & 0 & 0 \\ -\dot{\theta}_y & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot{\theta}_y \dot{\theta}_z \\ 0 & 0 & 0 \end{pmatrix}$$

giving

$$\sum \dot{s}_i \dot{s}_j \Big|_0 = \begin{pmatrix} 0 & \dot{\theta}_x \dot{\theta}_y & \dot{\theta}_x \dot{\theta}_z \\ 0 & 0 & \dot{\theta}_y \dot{\theta}_z \\ 0 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \dot{\theta}_{xy} & \dot{\theta}_{xz} \\ 0 & 0 & \dot{\theta}_{yz} \\ 0 & 0 & 0 \end{pmatrix} \text{ compressing notation.}$$

We are ready to reduce

$$\frac{d^2}{dt^2}\left[s\left(\vec{\theta}(t_0)\right)\right] = \sum \ddot{\theta}_* \frac{d}{d\theta_*} s_* \bigg|_0 - \sum \dot{\theta}_*^2 \tilde{s}_*\Big|_0 + 2 \sum \dot{s}_i \dot{s}_j \Big|_0$$

The last sum will be divided into two parts and absorbed into the first two sums.

$$2\sum \dot{s}_i \dot{s}_j\Big|_0 = 2\begin{pmatrix} 0 & \dot{\theta}_{xy} & \dot{\theta}_{xz} \\ 0 & 0 & \dot{\theta}_{yz} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dot{\theta}_{xy} & \dot{\theta}_{xz} \\ \dot{\theta}_{xy} & 0 & \dot{\theta}_{yz} \\ \dot{\theta}_{xz} & \dot{\theta}_{yz} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \dot{\theta}_{xy} & \dot{\theta}_{xz} \\ -\dot{\theta}_{xy} & 0 & \dot{\theta}_{yz} \\ -\dot{\theta}_{xz} & -\dot{\theta}_{yz} & 0 \end{pmatrix}$$

The new first sum becomes:

$$\sum \ddot{\theta}_* \frac{d}{d\theta_*} s_*\Big|_0 + \begin{pmatrix} 0 & \dot{\theta}_{xy} & \dot{\theta}_{xz} \\ -\dot{\theta}_{xy} & 0 & \dot{\theta}_{yz} \\ -\dot{\theta}_{xz} & -\dot{\theta}_{yz} & 0 \end{pmatrix} = [\vec{\ddot{\theta}}\times] + \begin{pmatrix} 0 & \dot{\theta}_{xy} & \dot{\theta}_{xz} \\ -\dot{\theta}_{xy} & 0 & \dot{\theta}_{yz} \\ -\dot{\theta}_{xz} & -\dot{\theta}_{yz} & 0 \end{pmatrix} = [\vec{\alpha}\times]$$

where $\vec{\alpha} \equiv \left(\ddot{\theta}_x - \dot{\theta}_y \dot{\theta}_z \text{ , } \ddot{\theta}_y + \dot{\theta}_x \dot{\theta}_z \text{ , } \ddot{\theta}_z - \dot{\theta}_x \dot{\theta}_y\right)$

The $\dot{\theta}_i \dot{\theta}_j$ terms contribute to the changing direction of $\vec{\omega}$

The new second sum becomes:

$$-\sum \dot{\theta}_*^2 \tilde{s}_*\Big|_0 + \begin{pmatrix} 0 & \dot{\theta}_{xy} & \dot{\theta}_{xz} \\ \dot{\theta}_{xy} & 0 & \dot{\theta}_{yz} \\ \dot{\theta}_{xz} & \dot{\theta}_{yz} & 0 \end{pmatrix} = \begin{pmatrix} -\dot{\theta}_y^2 - \dot{\theta}_z^2 & \dot{\theta}_{xy} & \dot{\theta}_{xz} \\ \dot{\theta}_{xy} & -\dot{\theta}_x^2 - \dot{\theta}_z^2 & \dot{\theta}_{yz} \\ \dot{\theta}_{xz} & \dot{\theta}_{yz} & -\dot{\theta}_x^2 - \dot{\theta}_y^2 \end{pmatrix} = [\vec{\omega}\times]^2$$

So $\frac{d^2}{dt^2}\left[s\left(\vec{\theta}(t_0)\right)\right] = [\vec{\alpha}\times] + [\vec{\omega}\times]^2$

Appendix H: Torque Free bounds on ω_{s3}

We consider the case where $L^2 - 2K\lambda_2 > 0$ and $\vec{\omega}$ initially satisfies

$$\lambda_1 \frac{\omega_1^2}{\omega_3^2} + \lambda_2 \frac{\omega_2^2}{\omega_3^2} < \epsilon_L$$

We will show that any subsequent point satisfies

$$\lambda_1 \frac{\omega_{s1}^2}{\omega_{s3}^2} + \lambda_2 \frac{\omega_{s2}^2}{\omega_{s3}^2} \le C\epsilon_L \quad \text{where } C \text{ is a constant.}$$

Any subsequent angular velocity $\vec{\omega}_s$ projects onto the hyperbola

$$\lambda_3(\lambda_3 - \lambda_2)\omega_{s3}^2 - \lambda_1(\lambda_2 - \lambda_1)\omega_{s1}^2 = L^2 - 2K\lambda_2$$

and we have $\dfrac{L^2 - 2K\lambda_2}{\lambda_3 - \lambda_2} \le \lambda_3 \omega_{s3}^2$

so that $\lambda_1 \omega_{s1}^2 + \lambda_2 \omega_{s2}^2 = 2K - \lambda_3 \omega_{s3}^2 \le 2K - \dfrac{L^2 - 2K\lambda_2}{\lambda_3 - \lambda_2}$

We compute a bound on this expression from the initial condition

$$\lambda_1 \omega_{s1}^2 + \lambda_2 \omega_{s2}^2 \le 2K - \frac{L^2 - 2K\lambda_2}{\lambda_3 - \lambda_2}$$

$$= \frac{2K(\lambda_3 - \lambda_2) - (L^2 - 2K\lambda_2)}{\lambda_3 - \lambda_2} = \frac{\lambda_3 2K - L^2}{\lambda_3 - \lambda_2} = \frac{\lambda_2(\lambda_3 - \lambda_2)\omega_2^2 + \lambda_1(\lambda_3 - \lambda_1)\omega_1^2}{\lambda_3 - \lambda_2}$$

$$= \lambda_2 \omega_2^2 + \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\lambda_1 \omega_1^2 \le \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\left(\lambda_2 \omega_2^2 + \lambda_1 \omega_1^2\right)$$

Finally

$$\lambda_1 \frac{\omega_{s1}^2}{\omega_{s3}^2} + \lambda_2 \frac{\omega_{s2}^2}{\omega_{s3}^2} \le \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\left(\lambda_2 \frac{\omega_2^2}{\omega_3^2} + \lambda_1 \frac{\omega_1^2}{\omega_3^2}\right)\frac{\omega_3^2}{\omega_{s3}^2}$$

$$\le \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\ \epsilon_L\ \left(\frac{2K}{\lambda_3}\right)\left(\frac{\lambda_3 - \lambda_2}{\lambda_3(L^2 - 2K\lambda_2)}\right) = \frac{2K}{\lambda_3^2(L^2 - 2K\lambda_2)}\ \epsilon_L$$

Appendix I: Summary of Derivatives of Angular Velocity

When the coordinate system is aligned with the principle axes,

$$\vec{L}=(\lambda_1\omega_1,\lambda_2\omega_2,\lambda_3\omega_3)$$

Using $\vec{0}=I\vec{\alpha}+\vec{\omega}\times I\vec{\omega}$ we can calculate $\vec{\alpha}$.

$$\begin{aligned}\vec{\alpha}=I^{-1}(\vec{I}\,\vec{\omega}\times\omega)&=I^{-1}((\lambda_1\omega_1,\lambda_2\omega_2,\lambda_3\omega_3)\times(\omega_1,\omega_2,\omega_3))\\&=I^{-1}((\lambda_2-\lambda_3)\omega_2\omega_3,(\lambda_3-\lambda_1)\omega_1\omega_3,(\lambda_1-\lambda_2)\omega_1\omega_2)\\&=\left(-\frac{(\lambda_3-\lambda_2)}{\lambda_1}\omega_2\omega_3\ ,\ \frac{(\lambda_3-\lambda_1)}{\lambda_2}\omega_1\omega_3\ ,\ -\frac{(\lambda_2-\lambda_1)}{\lambda_3}\omega_1\omega_2\right)\\&=(\dot{\omega}_1-\omega_2\omega_3\ ,\ \dot{\omega}_2+\omega_1\omega_3\ ,\ \dot{\omega}_3-\omega_1\omega_2)\end{aligned}$$

For the unit vector on the axis of rotation $\vec{r}_\omega\equiv\hat{\omega}$ we can calculate its acceleration as

$$\ddot{\vec{r}}_\omega=\vec{\omega}\times(\vec{\omega}\times\vec{r}_\omega)+\vec{\alpha}\times\ddot{\vec{r}}_\omega=\vec{\alpha}\times\vec{r}_\omega$$

$$=\frac{1}{\omega}\begin{pmatrix}-\frac{(\lambda_3-\lambda_2)}{\lambda_1}\omega_2\omega_3 & \frac{(\lambda_3-\lambda_1)}{\lambda_2}\omega_1\omega_3 & -\frac{(\lambda_2-\lambda_1)}{\lambda_3}\omega_1\omega_2\\ \omega_1 & \omega_2 & \omega_3\\ \hat{e}_1 & \hat{e}_2 & \hat{e}_3\end{pmatrix}$$

$$\begin{aligned}=\Bigg(&\frac{(\lambda_3-\lambda_1)}{\lambda_2\omega}\omega_1\omega_3^2+\frac{(\lambda_2-\lambda_1)}{\lambda_3\omega}\omega_1\omega_2^2\ ,\\&\frac{(\lambda_3-\lambda_2)}{\lambda_1\omega}\omega_2\omega_3^2-\frac{(\lambda_2-\lambda_1)}{\lambda_3\omega}\omega_1^2\omega_2\ ,\\&-\frac{(\lambda_3-\lambda_2)}{\lambda_1\omega}\omega_2^2\omega_3-\frac{(\lambda_3-\lambda_1)}{\lambda_2\omega}\omega_1^2\omega_3\Bigg)\end{aligned}$$

For the unit vector on the axis associated with λ_3 denoted as $\vec{r}_3=\hat{e}_3$,we can calculate its acceleration as

$$\ddot{\vec{r}}_3=\vec{\omega}\times(\vec{\omega}\times\vec{r}_3)+\vec{\alpha}\times\vec{r}_3$$

$$=\vec{\omega}\times\begin{pmatrix}\omega_1 & \omega_2 & \omega_3\\ 0 & 0 & 1\\ \hat{e}_1 & \hat{e}_2 & \hat{e}_3\end{pmatrix}$$

$$+\begin{pmatrix}-\frac{(\lambda_3-\lambda_2)}{\lambda_1}\omega_2\omega_3 & \frac{(\lambda_3-\lambda_1)}{\lambda_2}\omega_1\omega_3 & -\frac{(\lambda_2-\lambda_1)}{\lambda_3}\omega_1\omega_2\\ 0 & 0 & 1\\ \hat{e}_1 & \hat{e}_2 & \hat{e}_3\end{pmatrix}$$

$$=\begin{pmatrix}\omega_1 & \omega_2 & \omega_3\\ \omega_2 & -\omega_1 & 0\\ \hat{e}_1 & \hat{e}_2 & \hat{e}_3\end{pmatrix}+\left(\frac{(\lambda_3-\lambda_1)}{\lambda_2}\omega_1\omega_3\ ,\ \frac{(\lambda_3-\lambda_2)}{\lambda_1}\omega_2\omega_3\ ,\ 0\right)$$

$$=\left(\omega_1\omega_3\ ,\ \omega_2\omega_3\ ,\ -\omega_1^2-\omega_2^2\right)+\left(\frac{(\lambda_3-\lambda_1)}{\lambda_2}\omega_1\omega_3\ ,\ \frac{(\lambda_3-\lambda_2)}{\lambda_1}\omega_2\omega_3\ ,\ 0\right)$$

$$=\left(\frac{(\lambda_3+\lambda_2-\lambda_1)}{\lambda_2}\omega_1\omega_3\ ,\ \frac{(\lambda_3+\lambda_1-\lambda_2)}{\lambda_1}\omega_2\omega_3\ ,\ -\omega_1^2-\omega_2^2\right)$$

For the case: $\lambda_1=\lambda_2$ we get simplified terms resulting from $\lambda_1-\lambda_2=0$

$$\ddot{\vec{r}}_{\omega}=\frac{1}{\lambda_1\omega}\left((\lambda_3-\lambda_1)\omega_1\omega_3^2\ ,\ (\lambda_3-\lambda_1)\omega_2\omega_3^2\ ,\ -(\lambda_3-\lambda_1)\left(\omega_2^2\omega_3+\omega_1^2\omega_2\right)\right)$$
$$=\frac{(\lambda_3-\lambda_1)}{\lambda_1\omega}\left(\omega_1\omega_3^2\ ,\ \omega_2\omega_3^2\ ,\ -\left(\omega_2^2\omega_3+\omega_1^2\omega_2\right)\right)$$

$$\ddot{\vec{r}}_3=\left(\frac{\lambda_3}{\lambda_1}\omega_1\omega_3\ ,\ \frac{\lambda_3}{\lambda_1}\omega_2\omega_3\ ,\ -\omega_1^2-\omega_2^2\right)$$

For the case: $\lambda_1=\lambda_2=\lambda_3$, all the terms in $\vec{\alpha}$ contain differences in eigenvalues as factors. So
$$\vec{\alpha}=\vec{0}$$

Appendix J: Math Summary

Parametrize SO_3 from R^3 : $\vec{\theta} \to [s(\vec{\theta})]$ as:

$$[s(\vec{\theta})] = \begin{pmatrix} a_z & -b_z & 0 \\ b_z & a_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_y & 0 & b_y \\ 0 & 1 & 0 \\ -b_y & 0 & a_y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_x & -b_x \\ 0 & b_x & a_x \end{pmatrix}$$

Where the a_*'s and b_*'s are cos's and sin's of θ_*'s .

We get $\vec{v} = \dot{\vec{r}} = \frac{d}{dt}\left[s\left(\vec{\theta}(t)\right)\right]\vec{r} = \dot{\vec{\theta}} \times \vec{r} = \vec{\omega} \times \vec{r}$ for $\vec{\theta}(t) = \vec{0}$

$$\vec{a} = \ddot{\vec{r}} = \frac{d^2}{dt^2}\left[s\left(\vec{\theta}(t)\right)\right]\vec{r} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\alpha} \times \vec{r}$$

where $\vec{\alpha} \equiv \left(\ddot{\theta}_x - \dot{\theta}_y \dot{\theta}_z \, , \, \ddot{\theta}_y + \dot{\theta}_x \dot{\theta}_z \, , \, \ddot{\theta}_z - \dot{\theta}_x \dot{\theta}_y\right)$

Starting with $\vec{F} = m\ddot{\vec{r}}$, get $\vec{\tau} \equiv \vec{r} \times \vec{F} = \vec{r} \times m\ddot{\vec{r}} = -m\vec{r} \times (\vec{r} \times \vec{\alpha}) - m\vec{\omega} \times \left(\vec{r} \times (\vec{r} \times \vec{\omega})\right)$

Define the matrix

$$\mathtt{I} \equiv -m[\vec{r}\times]^2 = m\begin{pmatrix} y^2+z^2 & -xy & -xr_z \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$ and its integral version

to get

$$\vec{\tau} = \mathtt{I}\vec{\alpha} + \vec{\omega} \times \mathtt{I}\vec{\omega} \qquad\qquad \mathtt{I}(T\vec{r}) = T\ \mathtt{I}(\vec{r})\ T^{-1}$$

Define $\vec{L} \equiv \mathtt{I}\,\vec{\omega}$, then $\vec{\tau} = \frac{d}{dt}\vec{L}$

Also $K_{rot} = \frac{1}{2}\vec{\omega}\,\mathtt{I}\,\vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$, $\frac{d}{dt}K_{rot} = \vec{\tau}\cdot\vec{\omega}$ and $dK_{rot} = \vec{\tau}\cdot d\vec{\theta}$

$$\lambda_1 \le \frac{L^2}{2K} \le \lambda_3 \qquad \vec{\alpha} = -\mathtt{I}^{-1}(\vec{\omega} \times \vec{L}) = -\mathtt{I}^{-1}(\vec{\omega} \times \mathtt{I}\,\vec{\omega}) \qquad \left(\frac{d}{dt}\mathtt{I}\right)\vec{\omega} = \vec{\omega} \times \vec{L}$$

Appendix K: Reference from Goldstein

This is a quote from the references from Goldstein at the end of the chapter on the equations of motion of a rigid body, chap 5 pg . 179-180. It indicates the large scope of rigid body topics

F. KLEIN AND A. SOMMERFELD, *Theorie des Kreisels.* This monumental work on the theory of the top, in four volumes, has all the external appearances of the typical stolid and turgid German "Handbuch." Appearances are deceiving, however, for it is remarkably readable, despite the handicap of being written in the German language. The graceful, informal style has the fluency and attention to pedagogic details characteristic of all of Sommerfeld's later writings. Although the treatment becomes highly mathematical at times, the physical world is never lost sight of, and one does not founder in a maze of formula. Although limited by the title to tops and gyroscopes, the treatise actually provides a liberal education in all of rigid body mechanics, with excursions into other branches of physics and mathematics. Thus Chapter I discusses, among other items, Euler angles, infinitesimal rotations, and the Cayley-Klein parameters and their connections with the homographic transformation and with the theory of quaternions. The later notes to this chapter (in Vol. IV) discuss also the connections with electrodynamics and special relativity (quantum mechanics was still far in the future). By and large, Vol. I lays the necessary foundations in rigid dynamics and gives a physical description of top motion with little mathematics.

Vol. II is devoted to the detailed exposition of the heavy symmetrical top, although there is also much on Poinsot motion, and it contains a summary of what was then known about the asymmetric top. The distinction between regular and pseudoregular precession was first introduced here and the authors spend much time in examining the two motions, and the approach to regular precession. Many pages are given to a thorough demolishing of the popular or elementary "derivations" of gyroscopic precession. (The authors remark that it was the unsatisfactory nature of these derivations that led them to write the treatise!) There is a long discussion on questions of the stability of motion. Most of the treatment is based on the solution in terms of elliptic integrals and not merely on the approximate small nutation, as was done here.

Vol. III is mainly on perturbing forces (chiefly friction) and astronomical applications (nutation of the earth, precession of the equinoxes, etc.). The discussion of the wandering of the earth's poles is especially complete, including an estimation of the effects of the earth's elasticity and the transport of atmospheric masses by the wind circulation. Vol. IV is on technical applications,