Deriving the Maxwell Stress-Energy Tensor

Prologue

The expression of the Maxwell stress-energy tensor in terms of the Faraday force tensor has been in common use since the mid twentieth century. I felt compelled to tell a story that gives the illusion of a clear mathematical path that anyone would follow when discovering the Maxwell stress-energy tensor. For some engaging physical insight around this topic, consider Misner, Thorne and Wheeler's book on Gravitation.

This presentation might be of interest to physics students who have been exposed to the classical derivation of the Maxwell stress-energy tensor and have a hunger for the mathematical unity of electrodynamics expressed in terms of differential forms. This might also be of interest to mathematics students studying differential geometry. It shows how differential forms are useful to physicists.

This is an "onion skin" style of presentation where the main line of thinking is in the body, with details in appendices.

Introduction

We will be examining the 4x4 Maxwell stress-energy tensor in terms of the doubly covariant 4x4 electric Faraday field tensor F.

We are guided by the traditional 3x3 Maxwell stress tensor that is used for calculating force volume density in terms of the electric and magnetic fields and their derivatives. Historically, it is derived by starting with the Lorentz force law involving the fields together with the charge and current density. Then Maxwell's equations are used to replace the charge and current density with derivatives of the fields. After applying some tricky vector calculus identities we wind up with the Maxwell stress tensor where the divergence of rows give the components of force density, in terms of the fields and their derivatives.

The Maxwell stress-energy tensor is especially useful in the context of general relativity. Instead of using vector calculus, we will use properties of differential forms, exterior derivatives and the Hodge star. Many of these properties are special to 4-d space-time and the Minkowski metric.

$$g_{ij} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Starting with Force Density

Using Einstein notation, we can express the Lorentz force law as $f_i = F_{ij} J^j$. Here the doubly covariant electromagnetic Faraday field tensor F is expressed in matrix notation as:

$$F_{ij} \equiv \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

This differential form (of order two) can be expressed in wedge notation as:

$$F = -E_x\, dt \wedge dx - E_y\, dt \wedge dy - E_z\, dt \wedge dz + B_z\, dx \wedge dy - B_y\, dx \wedge dz + B_x\, dy \wedge dz$$

The contravariant vector $J^j = \begin{pmatrix} \rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$ is the charge-current density.

The time component of covariant force is classical power loss $f_t = -\vec{E} \cdot \vec{J}$

Maxwell's Equations

Classical Maxwell's equations can be expressed as:

$$\nabla \cdot \vec{B} = 0 \quad , \quad \nabla \times \vec{E} + \dot{\vec{B}} = \vec{0} \quad , \quad \nabla \cdot \vec{E} = \mu_0 \rho c^2 = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \vec{B} - \frac{1}{c^2} \dot{\vec{E}} = \mu_0 \vec{J}$$

We will express them as:

$$dF = 0 \qquad \text{and} \qquad *d*F = \mu_0 J_b$$

The charge-current density vector J is expressed in its lower index form, making it a differential form of order one. $J_b = \left(-\rho c^2, J_x, J_y, J_z\right)$

Here, d is the exterior derivative. The Hodge star operation $(*)$ depends on the Minkowski metric and calculates as:

$$*dt = -1/c\ dx \wedge dy \wedge dz$$
$$*dx = -c\, dt \wedge dy \wedge dz$$
$$*dy = c\, dt \wedge dx \wedge dz$$
$$*dz = -c dt \wedge dx \wedge dy$$

$$*(dt \wedge dx) = -1/c\ dy \wedge dz$$
$$*(dt \wedge dy) = 1/c\ dx \wedge dz$$
$$*(dt \wedge dz) = -1/c\ dx \wedge dy$$
$$*(dx \wedge dy) = c\, dt \wedge dz$$
$$*(dx \wedge dz) = -c\, dt \wedge dy$$
$$*(dy \wedge dz) = c\, dt \wedge dx$$

$$*(dx \wedge dy \wedge dz) = -c\, dt$$
$$*(dt \wedge dy \wedge dz) = -1/c\ dx$$
$$*(dt \wedge dx \wedge dz) = 1/c\ dy$$
$$*(dt \wedge dx \wedge dy) = -1/c\ dz$$

$$*(dt \wedge dx \wedge dy \wedge dz) = -1/c \qquad *1 = c\, dt \wedge dx \wedge dy \wedge dz$$

The Goal

We seek an order two symmetric tensor where:

- it is doubly covariant
- $*d*$ of the rows are force
- the time derivative of the space coordinates of the top row represent force
- The t-t coordinate is energy density times c^2
- It is nice enough to be used as a basic solution to boundary valued problems.

The result will be that the Maxwell stress-energy tensor can be expressed as:

$$\sigma_{ij} = \frac{1}{2\mu_0}\left(\langle F_i, F_j\rangle + \langle (*F)_i, (*F)_j\rangle\right)$$

About $*d*$

For a 1-form α_i , think of $*d*$ as the covariant version of the divergence. It calculates as:

$$*d*\alpha = -\frac{\partial \alpha^i}{\partial x^i} = -\sum_i \frac{\partial \alpha_i}{\partial x^i}\langle dx^i, dx^i\rangle = -\sum_i \left(d(\alpha_i)\right)_i \langle dx^i, dx^i\rangle$$

Note that d of a scalar is the ordinary gradient.

The inner product expression for differential forms $\langle , \rangle$ depends on the Minkowski metric and is employed as an alternative way to express raising an index.

For the time variable it evaluates as $\langle dt, dt\rangle = -1/c^2$
For a space variable the evaluation is like $\langle dx, dx\rangle = 1$
In summary, the exterior derivative of a 1-form is obtained by raising its index and negating its ordinary divergence.

For a 2-form like F_{ij} , think of $*d*F$ as taking the divergence and curl of the fields expressed in F.

In general, $*d*$ results in a differential form with one less degree.
In Minkowski space, the i^{th} component of $*d*F$ calculates as

$$(*d*F)_i = \frac{\partial F_i{}^j}{\partial x^j} = \sum_j \left(dF_{ij}\right)_j \langle dx^j, dx^j\rangle$$

Note that the components of $*d*F$ are almost the same as $*d*$ of the rows of F.

$$(*d*F)_i = -*d*(F_i)$$

Here, F_i is the 1-form made from the i^{th} row of F.

Derivatives of the Fields are Proxies for the Stuff

We will use Maxwell's in-homogeneous equation $(*d*F)_k = \mu_0 J_k$ to express force without explicit reference to charge or current (the stuff).

$$f_i = F_{ij}J^j = \frac{1}{\mu_0}F_{ij}(*d*F)^j = \frac{1}{\mu_0}\langle F_i \,,\, *d*F\rangle = \frac{1}{\mu_0}F_i{}^k\frac{\partial F_k{}^r}{\partial x^r}$$

This looks like a term in the derivative of a product.

Products of F and Divergence

Given that, we want to take the derivative of a product and our product is symmetric lets look at $F_i{}^k F_{jk} = [\langle F_i, F_j\rangle]$. We are viewing the product as an array of inner products of rows of F.

We take the divergence of a row and see what happens.

$$*d*[\langle F_i,F_j\rangle]_i = -\sum_j \left(d[\langle F_i,F_j\rangle]_i\right)_j \langle dx^j,dx^j\rangle = -\sum_{jk}\left((dF_{ik})_j F_{jk} + F_{ik}(dF_{jk})_j\right)\langle dx^k,dx^k\rangle\langle dx^j,dx^j\rangle$$

$\dots$

$$= \sum_k \langle dF_{ik} \,,\, F_k\rangle\langle dx^k,dx^k\rangle + \langle F_i \,,\, *d*F\rangle$$ This almost expresses force.

In terms of the fields:

$$[\langle F_i,F_j\rangle] = \begin{bmatrix} E^2 & -E_yB_z+E_zB_y & E_xB_z-E_zB_x & -E_xB_y+E_yB_x \\ -E_yB_z+E_zB_y & -E_x^2/c^2+B_z^2+B_y^2 & -E_xE_y/c^2-B_yB_x & -E_xE_z/c^2-B_zB_x \\ E_xB_z-E_zB_x & -E_xE_y/c^2-B_yB_x & -E_y^2/c^2+B_z^2+B_x^2 & -E_yE_z/c^2-B_zB_y \\ -E_xB_y+E_yB_x & -E_xE_z/c^2-B_zB_x & -E_yE_z/c^2-B_zB_y & -E_z^2/c^2+B_y^2+B_x^2 \end{bmatrix}$$

The above tensor has E^2 where energy should be. What about B^2 ?

Hodge Star and B

One of the features of the Hodge star and F is that it reverses the roles of E and B.

$$*F = \begin{bmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & 1/cE_z & -1/cE_y \\ -cB_y & -1/cE_z & 0 & 1/cE_x \\ -cB_z & 1/cE_y & -1/cE_x & 0 \end{bmatrix}$$

Then

$$\left[\left\langle (*F)_i , (*F)_j \right\rangle\right] = \begin{bmatrix} c^2B^2 & E_zB_y - E_yB_z & -E_zB_x + E_xB_z & E_yB_x - E_xB_y \\ E_zB_y - E_yB_z & -B_x^2 + \left(E_z^2 + E_y^2\right)/c^2 & -B_xB_y - E_yE_x/c^2 & -B_xB_z - E_zE_x/c^2 \\ -E_zB_x + E_xB_z & -B_xB_y - E_yE_x/c^2 & -B_y^2 + \left(E_z^2 + E_x^2\right)/c^2 & -B_yB_z - E_zE_y/c^2 \\ E_yB_x - E_xB_y & -B_xB_z - E_zE_x/c^2 & -B_yB_z - E_zE_y/c^2 & -B_z^2 + \left(E_y^2 + E_x^2\right)/c^2 \end{bmatrix}$$

Considering Energy

Note energy density $T = \frac{1}{2\mu_0}\left(\frac{E^2}{c^2} + B^2\right)$

We would like $\sigma_{tt} = c^2 T$

This tells that we should set the coordinates of the stress-energy tensor to be :

$$\sigma_{ij} = \frac{1}{2\mu_0}\left(\left\langle F_i , F_j \right\rangle + \left\langle (*F)_i , (*F)_j \right\rangle\right)$$

Divergence of a Row

A row of sigma can be expressed as: $\sigma_i = \frac{1}{2\mu_0}\left(\left[\left\langle F_i , F_j \right\rangle\right]_i + \left[\left\langle (*F)_i , (*F)_j \right\rangle\right]_i\right)$

We wish to calculate: $*d*\sigma_i$

So far, we have:

$$*d*\left[\left\langle F_i , F_j \right\rangle\right]_i = \sum_k \left\langle dF_{ik} , F_k \right\rangle \left\langle dx^k , dx^k \right\rangle + \left\langle F_i , *d*F \right\rangle$$

Substituting $*F$ for F, we get:

$$*d*\left[\left\langle (*F)_i , (*F)_j \right\rangle\right]_i = \sum_k \left\langle d(*F)_{ik} , (*F)_k \right\rangle \left\langle dx^k , dx^k \right\rangle - \left\langle (*F)_i , *dF \right\rangle$$

Adding the two sums on the left is tricky. (see Appendix C: Row Detail)

But, it gives us what we want:

$$\sum_k \langle dF_{ik}, F_k\rangle\langle dx^k, dx^k\rangle + \sum_k \langle d(*F)_{ik}, (*F)_k\rangle\langle dx^k, dx^k\rangle = \langle F_i, *d*F\rangle - \langle (*F)_i, *dF\rangle$$

This results in obtaining a force component:

$$*d*\sigma_i = \frac{1}{2\mu_0}\left(2\langle F_i, *d*F\rangle - 2\langle (*F)_i, *dF\rangle\right) = \frac{1}{\mu_0}\langle F_i, *d*F\rangle = f_i$$

We're Done

We have found our tensor where $*d*\sigma_i = f_i$. Out of the continuum of tensors that satisfy this relation, we like this one because of the symmetry of E and B, and it goes to zero at large distances from all charge and current. Furthermore $*d*\sigma_i = 0$ where there is no charge or current.

To use this in the context of general relativity, include momentum and density on the first row and column. Then $*d*\sigma_i = 0$ everywhere. Raise both indecies for use with the Einstein field equations. This could be used in the context of a rotating neutron star with some extra protons.

Appendix A: Hodge Star

The Hodge star operation on a differential form is similar in spirit to finding an orthogonal subspace. You are invisibly using the star operation, when a vector field is used in the context of flux integration through a surface rather than line integration along a path.

The metric of the space is involved with the star operation because of its role in integration.

If F is a k-form in n-space, $*F$ is an (n-k)-form.

We are working in Minkowski space where the defining expression for the Hodge star is $\omega \wedge *\omega = \langle \omega, \omega \rangle \sqrt{g}\ dt \wedge dx \wedge dy \wedge dz$. This definition assures its linear and tensorial nature.

Here, $\sqrt{g} = c$ and the Minkowski metric is embedded in the definition of $\langle \omega, \omega \rangle$.

Hodge Star of a 3-Form

Consider a simple 3-form, $\omega = \omega_{ijk}\, dx^i \wedge dx^j \wedge dx^k$

Then, $\langle \omega, \omega \rangle = \omega_{ijk}^2 \langle dx^i, dx^i \rangle \langle dx^j, dx^j \rangle \langle dx^k, dx^k \rangle$

Choose the index p that is different from i,j,k.

Then $*\omega = \omega_{ijk} \langle dx^i, dx^i \rangle \langle dx^j, dx^j \rangle \langle dx^k, dx^k \rangle \epsilon_{ijkp} \sqrt{g}\, dx^p$

$$= \omega_{ijk} \frac{\langle dx^i, dx^i \rangle \langle dx^j, dx^j \rangle \langle dx^k, dx^k \rangle \langle dx^p, dx^p \rangle}{\langle dx^p, dx^p \rangle} \epsilon_{ijkp} \sqrt{g}\, dx^p = \omega_{ijk} \frac{\operatorname{sgn}(g)}{\langle dx^p, dx^p \rangle \sqrt{g}} \epsilon_{ijkp}\, dx^p$$

Here ϵ_{ijkp} is ± 1 according to the permutation of ijkp

In our case the signature of g is given by sgn(g) = -1.

Hodge Star of a 2-Form

We apply the definition of the Hodge star to a basic 2-form and get:
$*(dx^i \wedge dx^j) = \epsilon_{ijpq} \langle dx^i, dx^i \rangle \langle dx^j, dx^j \rangle \sqrt{g}\ dx^p \wedge dx^q$

Here we have chosen the two indecies p and q that are different from i and j. The order of choice of p and q can result in a flipped sign. This ambiguity is resolved by ϵ_{ijpq} .

$$*F = \begin{bmatrix} 0 & cF_{yz} & -cF_{xz} & cF_{xy} \\ -cF_{yz} & 0 & -1/cF_{tz} & 1/cF_{ty} \\ cF_{xz} & 1/cF_{tz} & 0 & -1/cF_{tx} \\ -cF_{xy} & -1/cF_{ty} & 1/cF_{tx} & 0 \end{bmatrix}$$

Note that any expression that is true for a 2-form F is also true by replacing F with $*F$, and reducing $**F$ to -F (in Minkowski space).

Denoting the Combinatorics of a 2-Form
We see that the star operation on a 2-form mixes the elements and introduces coefficients.

We denote this by: $(*F)_{ij} = \gamma_{ijpq}\, F_{pq}$
We also get that: $F_{ij} = \mathrm{sgn}(g)\gamma_{ijpq}\,(*F)_{pq}$

Apply our previous representation of $*F$ and get:
$$*\left(F_{pq}\, dx^p \wedge dx^q\right) = \epsilon_{pqij}\left\langle dx^p, dx^p\right\rangle\left\langle dx^q, dx^q\right\rangle\sqrt{g}\;\; F_{pq}\, dx^i \wedge dx^j = (*F)_{ij}\, dx^i \wedge dx^j$$

Then set: $$\gamma_{ijpq} = \epsilon_{ijpq}\left\langle dx^p, dx^p\right\rangle\left\langle dx^q, dx^q\right\rangle\sqrt{g} = \frac{\mathrm{sgn}(g)\;\epsilon_{ijpq}}{\left\langle dx^i, dx^i\right\rangle\left\langle dx^j, dx^j\right\rangle\sqrt{g}}$$

Note $\epsilon_{ijpq} = \epsilon_{pqij}$

Appendix B: Inner Products and Derivatives

For a 2-form F, there are similarities in the pattern of the terms in $*F$ and $*dF$. This is surprising because $*F$ is a 2-form and $*dF$ is a 1-form. The similarity in pattern manifests itself in an inner product expression:

$$\left\langle *dA, (*B)_i \right\rangle = \operatorname{sgn}(g) \sum_{\substack{jk \\ k\neq i}} (dA_{ij})_k B_{jk} \langle dx^k, dx^k \rangle \langle dx^j, dx^j \rangle + \sum_j \left(d(*A)_{ij}\right)_i \langle dx^i, dx^i \rangle (*B)_{ij} \langle dx^j, dx^j \rangle$$

This appendix derives the above expression.

Calculating the Co-Derivative of a 2-Form

We will be using expressions:

$$(*dF)_j = -\sum_k \left(d(*F)_{jk}\right)_k \langle dx^k, dx^k \rangle \quad \text{and} \quad (*d*F)_j = \sum_k (dF_{jk})_k \langle dx^k, dx^k \rangle$$

To prove these expressions we will show:

$$*\left((dF)_{ijk}\, dx^i \wedge dx^j \wedge dx^k\right) = -\sum_k \left(d(*F)_{pk}\right)_k \langle dx^k, dx^k \rangle dx^p$$

for a single component of $*dF$ assuming $i<j<k$

We have:

$$(dF)_{ijk} = \epsilon_{ijk} \frac{\partial F_{jk}}{\partial x^i} + \epsilon_{jik} \frac{\partial F_{ik}}{\partial x^j} + \epsilon_{kij} \frac{\partial F_{ij}}{\partial x^k}$$

$$*\left((dF)_{ijk}\, dx^i \wedge dx^j \wedge dx^k\right) = \left(\epsilon_{ijk} \frac{\partial F_{jk}}{\partial x^i} + \epsilon_{jik} \frac{\partial F_{ik}}{\partial x^j} + \epsilon_{kij} \frac{\partial F_{ij}}{\partial x^k} \right) \frac{\operatorname{sgn}(g)\ \epsilon_{ijkp}}{\langle dx^p, dx^p \rangle \sqrt{g}}\ dx^p$$

note: $\dfrac{\partial F_{jk}}{\partial x^i} = \operatorname{sgn}(g)\ \gamma_{jkpi} \dfrac{\partial (*F)_{pi}}{\partial x^i}$

$$= \left(\epsilon_{ijk} \gamma_{jkpi} \frac{\partial (*F)_{pi}}{\partial x^i} + \epsilon_{jik} \gamma_{ikpj} \frac{\partial (*F)_{pj}}{\partial x^j} + \epsilon_{kij} \gamma_{ijpk} \frac{\partial (*F)_{pk}}{\partial x^k} \right) \frac{\epsilon_{ijkp}}{\langle dx^p, dx^p \rangle \sqrt{g}}\ dx^p$$

$$= \left(\frac{\epsilon_{ijk}\, \epsilon_{pijk}}{\langle dx^j, dx^j \rangle \langle dx^k, dx^k \rangle} \frac{\partial (*F)_{pi}}{\partial x^i} + \frac{\epsilon_{jik}\, \epsilon_{pjik}}{\langle dx^i, dx^i \rangle \langle dx^k, dx^k \rangle} \frac{\partial (*F)_{pj}}{\partial x^j} + \frac{\epsilon_{kij}\, \epsilon_{pkij}}{\langle dx^i, dx^i \rangle \langle dx^j, dx^j \rangle} \frac{\partial (*F)_{pk}}{\partial x^k} \right)$$

$$\left(\frac{\operatorname{sig}(g)\ \epsilon_{ijkp}}{\langle dx^p, dx^p \rangle \sqrt{g}^{\,2}}\ dx^p \right)$$

Note: $\epsilon_{ijk}\,\epsilon_{pijk} = \epsilon_{jik}\,\epsilon_{pjik} = \epsilon_{kij}\,\epsilon_{pkij}$ and $(\epsilon_{ijk}\,\epsilon_{pijk})\,\epsilon_{ijkp} = -1$

$$= -\left(\frac{1}{\langle dx^j, dx^j\rangle\langle dx^k, dx^k\rangle}\frac{\partial(*F)_{pi}}{\partial x^i} + \frac{1}{\langle dx^i, dx^i\rangle\langle dx^k, dx^k\rangle}\frac{\partial(*F)_{pj}}{\partial x^j} + \frac{1}{\langle dx^i, dx^i\rangle\langle dx^j, dx^j\rangle}\frac{\partial(*F)_{pk}}{\partial x^k} \right)$$
$$\left(\frac{\mathrm{sig}(g)}{\langle dx^p, dx^p\rangle\sqrt{g}^2}\ dx^p \right)$$

$$= -\left(\langle dx^i, dx^i\rangle \frac{\partial(*F)_{pi}}{\partial x^i} + \langle dx^j, dx^j\rangle \frac{\partial(*F)_{pj}}{\partial x^j} + \langle dx^k, dx^k\rangle \frac{\partial(*F)_{pk}}{\partial x^k} \right) dx^p$$

$$= -\sum_k \left(d(*F)_{pk}\right)_k \langle dx^k, dx^k\rangle\, dx^p$$

Calculating the Inner Product of 2-Forms with the Co-Derivative

$$\langle *dA, (*B)_i\rangle = -\sum_j \left(\sum_k \left(d(*A)_{jk}\right)_k \langle dx^k, dx^k\rangle \right) (*B)_{ij} \langle dx^j, dx^j\rangle$$

$$= -\sum_j \left(\sum_{k\neq i} \left(d(*A)_{jk}\right)_k \langle dx^k, dx^k\rangle \right) (*B)_{ij} \langle dx^j, dx^j\rangle - \sum_j \left(d(*A)_{ji}\right)_i \langle dx^i, dx^i\rangle (*B)_{ij} \langle dx^j, dx^j\rangle$$

segregating the k=i terms facilitates Hodge star combinatorics

$$= -\sum_{jk} \left(dA_{ip}\right)_k B_{pk}\, \gamma_{jkip}\, \gamma_{ijpk} \langle dx^k, dx^k\rangle \langle dx^j, dx^j\rangle - \sum_j \left(d(*A)_{ji}\right)_i \langle dx^i, dx^i\rangle (*B)_{ij} \langle dx^j, dx^j\rangle$$

note: $\gamma_{jkip}\,\gamma_{ijpk} = \epsilon_{ipjk}\,\epsilon_{pkij}\,\mathrm{sgn}(g)\dfrac{\langle dx^p, dx^p\rangle\langle dx^k, dx^k\rangle}{\langle dx^j, dx^j\rangle\langle dx^k, dx^k\rangle}$

$$= \mathrm{sgn}(g) \sum_{\substack{pk\\k\neq i}} \left(dA_{ip}\right)_k B_{pk} \langle dx^k, dx^k\rangle \langle dx^p, dx^p\rangle - \sum_j \left(d(*A)_{ji}\right)_i \langle dx^i, dx^i\rangle (*B)_{ij} \langle dx^j, dx^j\rangle$$

Rewrite, denoting the p index with j:

$$\langle *dA, (*B)_i\rangle = \mathrm{sgn}(g) \sum_{\substack{jk\\k\neq i}} \left(dA_{ij}\right)_k B_{jk} \langle dx^k, dx^k\rangle \langle dx^j, dx^j\rangle + \sum_j \left(d(*A)_{ij}\right)_i \langle dx^i, dx^i\rangle (*B)_{ij} \langle dx^j, dx^j\rangle$$

We also get:

$$-\langle *d*A, B_i\rangle = \mathrm{sgn}(g) \sum_{\substack{jk\\k\neq i}} \left(d(*A)_{ij}\right)_k (*B)_{jk} \langle dx^k, dx^k\rangle \langle dx^j, dx^j\rangle + \sum_j \left(dA_{ij}\right)_i \langle dx^i, dx^i\rangle B_{ij} \langle dx^j, dx^j\rangle$$

Appendix C: Row Detail

A row of the Maxwell stress-energy tensor can be expressed as:

$$\sigma_i = \frac{1}{2\mu_0}\left(\left[\langle F_i, F_j\rangle\right]_i + \left[\langle (*F)_i, (*F)_j\rangle\right]_i\right)$$

We wish to calculate: $*d*\left(\left[\langle F_i, F_j\rangle\right]_i + \left[\langle (*F)_i, (*F)_j\rangle\right]_i\right)$

Consider the first term:

$$*d*\left[\langle F_i, F_j\rangle\right]_i = -\sum_j \left(d\left[\langle F_i, F_j\rangle\right]_i\right)_j \langle dx^j, dx^j\rangle = -\sum_{jk}\left(\left(dF_{ik}\right)_j F_{jk} + F_{ik}\left(dF_{jk}\right)_j\right)\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle$$

$$= \sum_{jk}\left(dF_{ik}\right)_j F_{kj}\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle + \sum_{jk} F_{ik}\left(dF_{kj}\right)_j\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle$$

$$= \sum_{jk}\left(dF_{ik}\right)_j F_{kj}\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle + \langle F_i\, ,\, *d*F\rangle$$

After rearranging the terms in a way that reverses the roles of j and k we have:

$$*d*\left[\langle F_i, F_j\rangle\right]_i = \sum_{jk}\left(dF_{ij}\right)_k F_{jk}\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle + \langle F_i\, ,\, *d*F\rangle$$

and

$$*d*\left[\langle (*F)_i, (*F)_j\rangle\right]_i = \sum_{jk}\left(d(*F)_{ij}\right)_k (*F)_{jk}\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle - \langle (*F)_i\, ,\, *dF\rangle$$

The first sum shows a resemblance to:

$$\langle *dF, (*F)_i\rangle = -\sum_{\substack{jk\\ k\neq i}}\left(dF_{ij}\right)_k F_{jk}\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle + \sum_j \left(d(*F)_{ij}\right)_i (*F)_{ij}\langle dx^j, dx^j\rangle\langle dx^i, dx^i\rangle$$

(see Appendix B: Inner Products and Derivatives)

The first sum can be rewritten as:

$$\sum_{jk}\left(dF_{ij}\right)_k F_{jk}\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle$$

$$= -\langle *dF, (*F)_i\rangle + \sum_j \left(d(*F)_{ij}\right)_i (*F)_{ij}\langle dx^j, dx^j\rangle\langle dx^i, dx^i\rangle + \sum_j \left(dF_{ij}\right)_i F_{ji}\langle dx^i, dx^i\rangle\langle dx^j, dx^j\rangle$$

$$= -\langle *dF, (*F)_i\rangle - \sum_j \left(d(*F)_{ij}\right)_i (*F)_{ji}\langle dx^j, dx^j\rangle\langle dx^i, dx^i\rangle + \sum_j \left(dF_{ij}\right)_i F_{ji}\langle dx^i, dx^i\rangle\langle dx^j, dx^j\rangle$$

Finally we have:

$$\sum_{jk}\left(dF_{ij}\right)_k F_{jk}\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle + \sum_{jk}\left(d(*F)_{ij}\right)_k (*F)_{jk}\langle dx^k, dx^k\rangle\langle dx^j, dx^j\rangle$$

$$= -\langle *dF, (*F)_i\rangle - \sum_j \left(d(*F)_{ij}\right)_i (*F)_{ji}\langle dx^j, dx^j\rangle\langle dx^i, dx^i\rangle + \sum_j \left(dF_{ij}\right)_i F_{ji}\langle dx^i, dx^i\rangle\langle dx^j, dx^j\rangle$$
$$+\langle *d*F, F_i\rangle - \sum_j \left(dF_{ij}\right)_i F_{ji}\langle dx^j, dx^j\rangle\langle dx^i, dx^i\rangle + \sum_j \left(d(*F)_{ij}\right)_i (*F)_{ji}\langle dx^i, dx^i\rangle\langle dx^j, dx^j\rangle$$

$$= \langle F_i, *d*F\rangle - \langle (*F)_i, *dF\rangle$$

So, $*d*\left(\left[\langle F_i, F_j\rangle\right]_i + \left[\langle (*F)_i, (*F)_j\rangle\right]_i\right) = 2\langle F_i\, ,\, *d*F\rangle - 2\langle (*F)_i\, ,\, *dF\rangle$

Appendix D: Interpreting with E's and B's

By examining the E and B field expressions in the stress-energy tensor, we can deduce some energy and force expressions in terms of field derivatives. We proceed by taking the exterior derivative (raised divergence) of rows and equating them to force. These expressions are normally derived by using 3-d vector calculus identities.

Consider the first row of σ. It expresses energy and terms of $\vec{E}\times\vec{B}$.

$$\sigma_t = -\frac{1}{2\mu_0}\left(-E^2-c^2B^2;\ 2\,\vec{E}\times\vec{B}\right) \qquad \text{and} \qquad *d*\sigma_t = f_t = -\vec{E}\cdot\vec{J}$$

Now calculate $*d*\sigma_t$ directly:

$$\text{raised } \sigma_t = -\frac{1}{2\mu_0}\begin{pmatrix} E^2/c^2+B^2 \\ 2\,\vec{E}\times\vec{B}\end{pmatrix} \text{ so that } \quad *d*\sigma_t = \frac{1}{2\mu_0}\left(\frac{\partial}{\partial t}\left(E^2/c^2+B^2\right)+2\nabla\cdot\left(\vec{E}\times\vec{B}\right)\right)$$

$$\text{to get } \quad \nabla\cdot\left(\vec{E}\times\vec{B}\right) = -\frac{1}{2}\frac{d}{dt}\left(B^2+\frac{E^2}{c^2}\right)-\mu_0\vec{E}\cdot\vec{J}$$

This relates the divergence of the pointing vector to time derivative of energy.

Consider a spacial row

$$\sigma_x = -\frac{1}{2\mu_0}\left(\langle F_x, F_j\rangle+\langle(*F)_x,(*F)_j\rangle\right)$$

$$=-\frac{1}{2\mu_0}\left(E_yB_z-E_zB_y\ ,\ \ E_x^2/c^2-B_z^2-B_y^2\ ,\ \ E_xE_y/c^2+B_yB_x\ ,\ \ E_xE_z/c^2+B_zB_x\right)$$
$$-\frac{1}{2\mu_0}\left(-E_zB_y+E_yB_z\ ,\ \ B_x^2-\left(E_z^2+E_y^2\right)/c^2\ ,\ \ B_xB_y+E_yE_x/c^2\ ,\ \ B_xB_z+E_zE_x/c^2\right)$$

$$=-\frac{1}{2\mu_0}\left(E_yB_z-E_zB_y\ ,\ \ E_x^2/c^2+B_x^2-B^2\ ,\ \ E_xE_y/c^2+B_yB_x\ ,\ \ E_xE_z/c^2+B_zB_x\right)$$
$$-\frac{1}{2\mu_0}\left(-E_zB_y+E_yB_z\ ,\ \ B_x^2-\left(E^2-E_x^2\right)/c^2\ ,\ \ B_xB_y+E_yE_x/c^2\ ,\ \ B_xB_z+E_zE_x/c^2\right)$$

We get raised $\sigma_x =$

$$-\frac{1}{2\mu_0}\begin{pmatrix} \left(-E_y B_z + E_z B_y\right)/c^2 \\ E_x^2/c^2 + B_x^2 - B^2 \\ E_x E_y/c^2 + B_y B_x \\ E_x E_z/c^2 + B_z B_x \end{pmatrix} - \frac{1}{2\mu_0}\begin{pmatrix} \left(E_z B_y - E_y B_z\right)/c^2 \\ B_x^2 - \left(E^2 - E_x^2\right)/c^2 \\ B_x B_y + E_y E_x/c^2 \\ B_x B_z + E_z E_x/c^2 \end{pmatrix} = -\frac{1}{2\mu_0}\begin{pmatrix} -2\left(\vec{E}\times\vec{B}\right)_x/c^2 \\ 2E_x^2/c^2 - E^2/c^2 + 2B_x^2 - B^2 \\ 2E_x E_y/c^2 + 2B_x B_y \\ 2E_x E_z/c^2 + 2B_x B_z \end{pmatrix}$$

Direct calculation of the divergence gives:

$$\begin{aligned} f_x &= *d*\sigma_x \\ &= \frac{1}{2\mu_0}\left(-\frac{\partial}{\partial t}2\left(\vec{E}\times\vec{B}\right)_x/c^2 - \frac{\partial}{\partial x}\left(E^2/c^2 + B^2\right) + 2\left(\left(\nabla E_x\right)\cdot\vec{E} + E_x\left(\nabla\cdot\vec{E}\right)\right)/c^2 + 2\left(\left(\nabla B_x\right)\cdot\vec{B} + B_x\left(\nabla\cdot\vec{B}\right)\right)\right) \end{aligned}$$

To get:

$$f_x = \epsilon_0\left[\left(\nabla\cdot\vec{E}\right)E_x + \vec{E}\cdot\nabla E_x\right] + \frac{1}{\mu_0}\left[\left(\nabla\cdot\vec{B}\right)B_x + \vec{B}\cdot\nabla B_x\right] - \frac{1}{2}\left[\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\right]_x - \epsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)_x$$

This relates force to the divergence of the fields, the gradient of energy, and the time derivative of the pointing vector.

Appendix E: Identities and Formulas

This is a collection of differential form identities that are useful when studying electrodynamics. Some of these were proved in other appendices.

For F a 2-form:

$$(*F)_{ij} = \gamma_{ijpq}\, F_{pq} \quad \text{with} \quad i \neq p \neq j \quad \text{and} \quad i \neq q \neq j$$

$$\gamma_{ijpq} = \epsilon_{ijpq} \langle dx^p, dx^p \rangle \langle dx^q, dx^q \rangle \sqrt{g} = \frac{\operatorname{sgn}(g)\, \epsilon_{ijpq}}{\langle dx^i, dx^i \rangle \langle dx^j, dx^j \rangle \sqrt{g}}$$

$$**F = \operatorname{sgn}(g)\, F$$

For α a 1-form and F a 2-form:

$$*d*\alpha = -\sum_i \frac{\partial \alpha_i}{\partial x^i} \langle dx^i, dx^i \rangle$$

$$(*d*F)_i = \sum_j \frac{\partial F_{ij}}{\partial x^j} \langle dx^j, dx^j \rangle$$

For 1-forms α and β , and 2-forms A and B:

$$\langle \alpha, \beta \rangle = \sum_i \alpha_i \beta_i \langle dx^i, dx^i \rangle$$

$$\langle A, B \rangle = \sum_{i<j} A_{ij} B_{ij} \langle dx^i, dx^i \rangle \langle dx^j, dx^j \rangle = \frac{1}{2} \sum_{ij} A_{ij} B_{ij} \langle dx^i, dx^i \rangle \langle dx^j, dx^j \rangle = \frac{1}{2} \sum_i \langle A_i, B_i \rangle \langle dx^i, dx^i \rangle$$

$$\langle (*A)_i, (*B)_j \rangle = -\operatorname{sgn}(g) \langle A_j, B_i \rangle \quad \text{for} \quad i \neq j$$

$$\langle (*A)_i, (*B)_i \rangle = \operatorname{sgn}(g) \left(\frac{\langle A, B \rangle}{\langle dx^i, dx^i \rangle} - \langle A_i, B_i \rangle \right)$$

For 2-forms A and B:

$$\langle *A, *B \rangle = \operatorname{sgn}(g) \langle A, B \rangle$$

$$\langle A, *B \rangle = -\operatorname{sgn}(g) \langle *A, B \rangle$$

$$\langle A_i, (*B)_j \rangle = \operatorname{sgn}(g) \langle (*A)_j, B_i \rangle \quad \text{for} \quad i \neq j$$

$$\langle A_i, (*B)_i \rangle = \operatorname{sgn}(g) \left(\langle (*A)_i, B_i \rangle - \frac{\langle *A, B \rangle}{\langle dx^i, dx^i \rangle} \right)$$

For F the Faraday field tensor:

$$dF = 0$$

$$*d*F = \mu_0 J_b$$

$$f_i = \frac{1}{\mu_0}\langle F_i\,,\,*d*F\rangle$$

$$\langle F_i, F_j\rangle + \langle (*F)_i, (*F)_j\rangle \;=\; 2\langle F_i, F_j\rangle - \frac{1}{2} g_{ij}\langle F, F\rangle$$

$$*d*\left(\left[\langle F_i, F_j\rangle\right]_i + \left[\langle (*F)_i, (*F)_j\rangle\right]_i\right) \;=\; 2\langle F_i\,,\,*d*F\rangle - 2\langle (*F)_i\,,\,*dF\rangle$$