E&M and the Faraday Field Tensor

Introduction

We all love Maxwell's equations:

$$\nabla\cdot\vec{B}=0 \quad , \quad \nabla\times\vec{E}+\dot{\vec{B}}=\vec{0} \quad , \quad \nabla\cdot\vec{E}=\mu_0\,\rho\, c^2=\frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla\times\vec{B}-\frac{1}{c^2}\dot{\vec{E}}=\mu_0\vec{J}$$

These equations state the relationship between electric and magnetic fields (E and B) and charge and current densities $\left(\rho \text{ and } \vec{J}\right)$. The divergence and curl $\left(\nabla\cdot \text{ and } \nabla\times\right)$ derivative operations are prominent here.

There is a compact and coordinate independent way of writing Maxwell's equations:

$$dF=0 \qquad \text{and} \qquad *d*F=\mu_0\vec{J}_b$$

I will explain why we write it this way and the meaning of F , d , and $*$. This topic is usually presented with $c=1$ and $\mu_0=1$. This journey will illuminate the relationship between special relativity and electrodynamics, and the role of the vector potential.

What is F ?

We begin by writing down the calculations required to calculate the combined electric and magnetic force density on an object with given charge and current densities. Inspired by matrix representation of a cross product, write the Lorenz force law in matrix notation as:

$$\vec{f}=\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}\begin{pmatrix} \rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \rightarrow \left(-\vec{E}\cdot\vec{J}\ ;\ \rho\vec{E}+\vec{J}\times\vec{B}\right)$$

The right three coordinates of the result give the force density vector, while the zeroth coordinate gives the power loss density. We will be associating this coordinate with time.

I have been vague about how I am using row and column vectors. I am going to change the above expression into tensor notation. Tensor notation is a way of writing vectors, matrices and multi-index arrays so that the rules for changing coordinate systems are clear, if you know the derivatives of the coordinate transformation.

Define the electromagnetic Faraday field tensor:

$$F_{ij} \equiv \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$ as a doubly covariant tensor.

F_{ij} acts on the four-current density vector $J^i = \vec{J} = \begin{pmatrix} \rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$

which expresses charge and current density.

The Lorentz force law can be expressed as $f_i = F_{ij} J^j$.

The resulting vector is covariant, which is why I originally wrote it as a row vector. The choice of covariant or contravarient representation needs to be done carefully, but ultimately it is a matter of convenience.

The benefit of writing the Lorentz force law in tensor notation is that, we know how to do the calculations in other coordinate systems, such as polar coordinates. The alternative is to figure out the expression for differential operators (such as the divergence) in the other coordinate system.

What is dF ?
I will explain dF after I describe differential forms and the exterior derivative.

Differential Forms
Differential forms are frequently used as infinitesimal expressions for calculating the area or volume spanned by a set of infinitesimal vectors. This value associated with a set of vectors can be represented as a covariant tensor. A differential form tensor is anti-symmetric because of its relationship with area and volume. Because of this symmetric redundancy, a wedge notation is often used.

A second order differential form acts on a pair of vectors to obtain a scalar (representing area). In this case we can abuse notation slightly, and represent it as a matrix. The action on a pair of vectors is denoted by writing one vector as a row on the left of the matrix and the other as a column vector on the right.

We will not be using infinitesimal thinking on differential forms. Instead, we will be using their special linearity properties.

In the case of F_{ij} , we see that it is a differential form because it is doubly covariant and anti-symmetric. We will remove the symmetric redundancy in our expression of F by changing to wedge notation. I will use the following graphic to help me write the terms:

$$\begin{bmatrix} 0 & dt\wedge dx & dt\wedge dy & dt\wedge dz \\ \cdot & 0 & dx\wedge dy & dx\wedge dz \\ \cdot & \cdot & 0 & dy\wedge dz \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

So, we can express F as:

$$F=-E_x\,dt\wedge dx - E_y\,dt\wedge dy - E_z\,dt\wedge dz + B_z\,dx\wedge dy - B_y\,dx\wedge dz + B_x\,dy\wedge dz$$

This will help us in later calculations because we can use wedge product rules like: $dx\wedge dx=0$ and $dx\wedge dy=-dy\wedge dx$

The Exterior Derivative

The exterior derivative of a differential form $(d\omega)$, results in a new form with a higher degree. The divergence and curl of a vector field are manifestations of the exterior derivative. The exterior derivative is an extension of the ordinary derivative that also extends the fundamental theorem of calculus to the general Stokes theorem. Also, if the exterior derivative is applied twice, the result is zero $(dd\omega=0)$, like $\text{curl grad}=\vec{0}$. The exterior derivative is a tensor, unlike the naive Jacobian matrix of a vector field. The role of Gauss' law and Stokes theorem in electrodynamics is a hint that the exterior derivative may be involved.

The calculation method for taking the exterior derivative involves taking all the partial derivatives and applying the wedge product rules. For example:

$$d\left(B_x\,dy\wedge dz\right)$$

$$=\frac{\partial B_x}{\partial t}dt\wedge dy\wedge dz + \frac{\partial B_x}{\partial x}dx\wedge dy\wedge dz + \frac{\partial B_x}{\partial y}dy\wedge dy\wedge dz + \frac{\partial B_x}{\partial z}dz\wedge dy\wedge dz$$

$$=\frac{\partial B_x}{\partial t}dt\wedge dy\wedge dz + \frac{\partial B_x}{\partial x}dx\wedge dy\wedge dz + 0 + 0$$

Similarly we calculate dF as:

$$dF=\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)dx\wedge dy\wedge dz+\left(\frac{\partial B_x}{\partial t}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)dt\wedge dy\wedge dz$$
$$+\left(-\frac{\partial B_y}{\partial t}+\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)dt\wedge dx\wedge dz+\left(\frac{\partial B_z}{\partial t}+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)dt\wedge dx\wedge dy$$

Maxwell's Homogeneous Equations

We now have a simple coordinate independent expression for two of Maxwell's equations.

Note that the expression $dF=0$ means that all the terms in the above expression are zero. Then we re-interpret the terms of dF and get:

$$\nabla\cdot\vec{B}=0 \quad \text{and} \quad \nabla\times\vec{E}+\dot{\vec{B}}=\vec{0}$$

So $dF=0$ expresses Maxwell's two Homogeneous equations.

Maxwell's Inhomogeneous Equations

Maxwell's inhomogeneous equations are $\nabla\cdot\vec{E}=\frac{\rho}{\epsilon_0}$ and $\nabla\times\vec{B}-\frac{1}{c^2}\dot{\vec{E}}=\mu_0\vec{J}$

They can also be expressed as $*d*F=\mu_0\vec{J}_b$ where $\vec{J}_b$ is the flattened (lowered index) four-current density.

To understand this, we will visit special relativity and the related Hodge star operation.

Special Relativity

An important way to study special relativity is through the Minkowski pseudo-metric on space-time:

$$g_{ij}=\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Typically, a metric is an extension of the idea of a dot product. So, it is a symmetric covariant tensor (not a differential form). Given the idea of a dot product, the square norm of a vector can be defined. Here, the Minkowski square norm of a vector is given by $-c^2t^2+x^2+y^2+z^2 \;=\; \|\vec{x}\|_M^2 \;=-c^2t^2+\|\vec{x}\|_3^2$.

Typically, the expression for g_{ij} and the algebra for calculating the norm of a vector will change, when you change coordinate systems. The Minkowski metric is special because the expression for the Minkowski norm does not change under a Lorentz transformation.

We demonstrate Lorentz invariance by examining a basic Lorentz transformation:

$$x'=\gamma(x-vt) \qquad t'=\gamma(t-vx/c^2)$$

or $$x'=\gamma(x-\beta ct) \qquad t'=\gamma(t-\beta x/c)$$

where $\vec{v}$ is in the $\hat{x}$ direction, $\gamma=\frac{1}{\sqrt{1-\beta^2}}$ and $\beta=\frac{v}{c}$ so that $\gamma^2\left(1-\beta^2\right)=1$

Also $y'=y$ and $z'=z$

The primed coordinates are stationary with respect to the moving coordinate system. They represent “proper” coordinates where v' = 0 .

We try the square norm expression on a transformed vector and find that the expression is unchanged:

$$-c^2(t')^2+(x')^2+(y')^2+(z')^2=-c^2(t\gamma-x\gamma\beta/c)^2+(-ct\beta\gamma+x\gamma)^2+y^2+z^2$$

$$=-c^2t^2\left(\gamma^2-\gamma^2\beta^2\right)+x^2\left(\gamma^2-\gamma^2\beta^2\right)+y^2+z^2=-c^2t^2+x^2+y^2+z^2$$

The invariance of the Minkowski norm will illuminate the invariance of Maxwell's equations under Lorenz transformations.

The Hodge Star Operation

The Hodge star operation on a differential form is similar in spirit to finding an orthogonal subspace. The metric of the space becomes involved because of the role of area and volume. You are invisibly using the star operation, when a vector field is used in the context of flux integration through a surface rather than line integration along a path.

With the Minkowski pseudo-metric there is an associated Hodge star operation on differential forms.

For our purposes, the Hodge star is defined for a differential form ω by
$\omega\wedge*\omega=\langle\omega,\omega\rangle\sqrt{|g_{ij}|}\,\det()$

In our case(the signature is -+++) $\omega\wedge*\omega=\langle\omega,\omega\rangle c(dt\wedge dx\wedge dy\wedge dz)$

$*dt=-1/c\ dx\wedge dy\wedge dz$
$*dx=-c\,dt\wedge dy\wedge dz$
$*dy=c\,dt\wedge dx\wedge dz$
$*dz=-c\,dt\wedge dx\wedge dy$

$*(dt\wedge dx)=-1/c\ dy\wedge dz$
$*(dt\wedge dy)=1/c\ dx\wedge dz$
$*(dt\wedge dz)=-1/c\ dx\wedge dy$
$*(dx\wedge dy)=c\,dt\wedge dz$
$*(dx\wedge dz)=-c\,dt\wedge dy$
$*(dy\wedge dz)=c\,dt\wedge dx$

$*(dx\wedge dy\wedge dz)=-c\,dt$
$*(dt\wedge dy\wedge dz)=-1/c\ dx$
$*(dt\wedge dx\wedge dz)=1/c\ dy$
$*(dt\wedge dx\wedge dy)=-1/c\ dz$

$*(dt\wedge dx\wedge dy\wedge dz)=-1/c$ $\quad *1=c\,dt\wedge dx\wedge dy\wedge dz$

Calculating $*d*F = \mu_0 \vec{J}_b$

We have been using $\vec{J}$ as a contravariant vector: $\vec{J} = \begin{pmatrix} \rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$.

In Euclidian space, a contravariant column vector can be changed to a covariant vector simply by writing it as a row vector, because of its trivial metric. With our given Minkowski pseudo-metric, we use the flattening operation, denoted by the musical flat symbol. It is defined by $\vec{J}_b = J_j = J^i g_{ij}$. This operation is also called lowering an index.

We have $\vec{J}_b = \left(-\rho c^2, J_x, J_y, J_z\right)$

Any covariant vector field can also by considered a differential form of order one because it can act on a vector field to produce a scalar field. Anti-symmetry is not an issue because the differential form only acts on one vector.

$\vec{J}_b = -\rho c^2 dt + J_x dx + J_y dy + J_z dz$

Before doing the calculation, note that:

F	is a 2-form
$*F$	is a 2-form (4 - 2 = 2)
$d*F$	is a 3-form
$*d*F$	is a 1-form (4 - 3 = 1)
$\mu_0 \vec{J}_b$	is a 1-form

Now calculate,

$$\begin{aligned} *F &= E_x/c\ dy\wedge dz - E_y/c\ dx\wedge dz + E_z/c\ dx\wedge dy + cB_z dt\wedge dz + cB_y dt\wedge dy + cB_x dt\wedge dx \\ &= cB_x dt\wedge dx + cB_y dt\wedge dy + cB_z dt\wedge dz + E_z/c\ dx\wedge dy - E_y/c\ dx\wedge dz + E_x/c\ dy\wedge dz \end{aligned}$$

$$\begin{aligned} d*F = {} & \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx\wedge dy\wedge dz + \left(\frac{1}{c}\frac{\partial E_x}{\partial t} - c\frac{\partial B_z}{\partial y} + c\frac{\partial B_y}{\partial z}\right) dt\wedge dy\wedge dz \\ & + \left(-\frac{1}{c}\frac{\partial E_y}{\partial t} - c\frac{\partial B_z}{\partial x} + c\frac{\partial B_x}{\partial z}\right) dt\wedge dx\wedge dz + \left(\frac{1}{c}\frac{\partial E_z}{\partial t} - c\frac{\partial B_y}{\partial x} + c\frac{\partial B_x}{\partial y}\right) dt\wedge dx\wedge dy \end{aligned}$$

$$\begin{aligned} *d*F = {} & -\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dt - \left(\frac{1}{c^2}\frac{\partial E_x}{\partial t} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z}\right) dx \\ & - \left(\frac{1}{c^2}\frac{\partial E_y}{\partial t} + \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) dy - \left(\frac{1}{c^2}\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y}\right) dz \end{aligned}$$

$$= -\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)dt+\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}-\frac{1}{c^2}\frac{\partial E_x}{\partial t}\right)dx$$
$$+\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}-\frac{1}{c^2}\frac{\partial E_y}{\partial t}\right)dy+\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}-\frac{1}{c^2}\frac{\partial E_z}{\partial t}\right)dz$$

Reinterpreting the terms, we get Maxwell's inhomogeneous equations:

$*d*F=\mu_0\vec{J}_b \quad \Rightarrow \quad \nabla\cdot\vec{E}=\mu_0\rho c^2=\frac{\rho}{\epsilon_0}$ and $\nabla\times\vec{B}-\frac{1}{c^2}\dot{\vec{E}}=\mu_0\vec{J}$ (and vice versa)

Invariance of Maxwell's Equations under Lorentz Transformations

In a naive view of Newtonian mechanics, laws like conservation of momentum, should remain invariant in moving coordinate systems as long as there is no acceleration (Galilean invariance). The laws of electricity and magnetism are not invariant under Galilean transformations as demonstrated by magnetic effects.

Our philosophy of physical invariance is rescued when we consider Lorentz transformations. Our tensorial expressions for Maxwell's equations gives us an easy way to show that the vector algebraic expressions for Maxwell's equations are identical in any Lorentz transformed coordinate systems. (Mechanics is also invariant after tweaking the definition of momentum.)

We have expressed Maxwell's equations in tensor form as

$$dF=0 \qquad \text{and} \quad *d*F=\mu_0\vec{J}_b$$

As tensor expressions, they remain true in different coordinate systems.

However, the expressions for F and the corresponding fields $\vec{E}$, $\vec{B}$, and $\vec{J}_b$ could change.

In the case of the homogeneous expressions, the algebra of the exterior derivative is the same in any coordinate system. So the homogeneous equations have the same expression regardless of coordinate system.

In the case of the inhomogeneous equations, the algebraic expression for the metric and the $*$ operation will change. So the algebraic expressions for the inhomogeneous Maxwell equations might also change. However, if the change in coordinate systems is a Lorentz transformation, the metric will be unchanged. So, the $*$ operation will be unchanged. This invariance propagates to the algebraic expressions of the inhomogeneous Maxwell equations.

Invariance of the Homogeneous Equations and Polar Coordinates

The Lorentz transformations are quite restrictive. It is plausible that Maxwell's equations have an unchanged representation in the transformed coordinates.

What about the homogeneous equations and more general transformations such as polar coordinates? For example, why not use the spherical divergence rather than the naive divergence defined by transcribing the Cartesian divergence?

The above contradiction is explained away by noting that the transformed electric and magnetic fields are transcribed from the transformed electromagnetic field tensor. In previous experience, you may have left the fields unchanged, but used transformed differential operators. In this presentation, it is the fields and differential forms that get transformed and the differential operators remain unchanged.

Why Differential Forms?

We have used the mathematics of differential forms to express Maxwell's equations. This formalism is a good choice to express the Physics of E&M because the divergence theorem and Green's theorem are embedded in the mathematics of differential forms. Also, we can encode special relativity via the Minkowski pseudo-metric.

The Four-Potential

Note that the electromagnetic Faraday field tensor F is a 2-form. We will consider the idea of having a 1-form $\Phi = -\phi\, dt + A_x dx + A_y dy + A_z dz$ with the property that $F = d\Phi$.

If this is the case then we automatically get $dF = dd\Phi = 0$.

But first, we mention the Poincaré lemma and Helmholtz' decomposition theorem.

Poincaré Lemma

A differential form ω is called closed if $d\omega = 0$.

A differential form ω is called exact if it is the exterior derivative of something, $\omega = d\alpha$

We automatically get that an exact form is closed from $d\omega = dd\alpha = 0$
The converse is the subject of the Poincaré lemma and is most easily proved using algebraic topology. For our purposes, the Poincaré lemma says that a closed form is exact if it is defined on a space without holes.

We illustrate this with a vector field having a zero curl. This (co-) vector field $\vec{v}$, represents our closed form ω . We know that the line integral around any loop is zero, inside a simple region. So, we can construct a real valued function f, on the simple region where the vector field $\vec{v}$ is the gradient of f . The scalar function f represents α in this discussion.

Consider the case of the circulating magnetic vector field around a thin wire. Away from the wire, it has a zero curl. However, there is a discontinuity at the wire, and we have to cut the wire out of our space of field definition. In this idealization, we cannot construct a scalar valued function whose gradient is the vector field. We can construct such a function if we restrict our view to a simple region (no holes). It looks like part of a helical ramp.

Helmholtz' Decomposition Theorem
This theorem roughly states that any vector field in R^3 can be expressed as the gradient of something plus the curl of something. We will use this to express electric and magnetic fields.

Finding The Four-Potential of F
We take courage from the Poincaré Lemma and inspiration from the Helmholtz' decomposition theorem and define a 1-form four-potential as
$\Phi = -\phi dt + A_x dx + A_y dy + A_z dz$, where ϕ and $\vec{A}$ are called scalar and vector potentials. This expression of Φ will lead to definitions of $\vec{E}$ and $\vec{B}$ that are consistent with $d\Phi = F$.

Calculate the exterior derivative of Φ as:

$$\begin{aligned}
d\Phi = &-\frac{\partial\phi}{\partial x}dx\wedge dt - \frac{\partial\phi}{\partial y}dy\wedge dt - \frac{\partial\phi}{\partial z}dz\wedge dt \\
&+\frac{\partial A_x}{\partial y}dy\wedge dx + \frac{\partial A_x}{\partial z}dz\wedge dx + \frac{\partial A_x}{\partial t}dt\wedge dx \\
&+\frac{\partial A_y}{\partial x}dx\wedge dy + \frac{\partial A_y}{\partial z}dz\wedge dy + \frac{\partial A_y}{\partial t}dt\wedge dy \\
&+\frac{\partial A_z}{\partial x}dx\wedge dz + \frac{\partial A_z}{\partial y}dy\wedge dz + \frac{\partial A_z}{\partial t}dt\wedge dz
\end{aligned}$$

$$\begin{aligned}
= &\left(\frac{\partial\phi}{\partial x}dt\wedge dx + \frac{\partial\phi}{\partial y}dt\wedge dy + \frac{\partial\phi}{\partial z}dt\wedge dz\right) \\
&+\left(\frac{\partial A_x}{\partial t}dt\wedge dx + \frac{\partial A_y}{\partial t}dt\wedge dy + \frac{\partial A_z}{\partial t}dt\wedge dz\right) \\
&+\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)dx\wedge dy + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)dx\wedge dz + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)dy\wedge dz
\end{aligned}$$

If the four-potential Φ exists, the E-M fields must satisfy $\vec{E} \equiv -\nabla\phi - \dot{\vec{A}}$ and $\vec{B} \equiv \nabla \times \vec{A}$ to get $F = d\Phi$. And of course we get $0 = dd\Phi = dF$.

Using Maxwell's inhomogeneous equations $*d*F = \mu_0 \vec{J}_b$, we can extract charge and current density. So, the four-vector potential Φ , gives a full description of an electrodynamic system.

Given the Poincaré Lemma, we can go the other way and know that there is a four-potential Φ for any closed 2-form F on R^4 , and extract fields analogous to $\vec{E}$ and $\vec{B}$ with an analogous four-vector $\vec{J}$ satisfying Maxwell's equations.

Special Relativity Summary
Electrodynamics is completly described by special ralativity via the Minkowski metric, the mathematics of differential forms and any four-potential.

Conservation of charge
Maxwell's inhomogeneous equations $*d*F = \mu_0 \vec{J}_b$ imply the conservation of charge.

In classical 3-d E&M charge conservation is expressed as $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$.
Having serendipitously chosen ρ as the temperal component of four-current, we can write conservation of charge as $*d*\vec{J}_b = 0$. We verify with

$$\vec{J} = \begin{pmatrix} \rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \qquad \vec{J}_b = \left(-c^2\rho, J_x, J_y, J_z\right) = -c^2\rho\, dt + J_x dx + J_y dy + J_z dz$$

$$*\vec{J}_b = c\rho\, dx \wedge dy \wedge dz - cJ_x dt \wedge dy \wedge dz + cJ_y dt \wedge dx \wedge dz - cJ_z dt \wedge dx \wedge dy$$

$$\begin{aligned} d*\vec{J}_b = {} & c\frac{\partial \rho}{\partial t} dt \wedge dx \wedge dy \wedge dz \\ & + c\frac{\partial J_x}{\partial x} dt \wedge dx \wedge dy \wedge dz + c\frac{\partial J_y}{\partial y} dt \wedge dx \wedge dy \wedge dz + c\frac{\partial J_z}{\partial z} dt \wedge dx \wedge dy \wedge dz \end{aligned}$$

$$*d*\vec{J}_b = -\left(\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right)$$

So, $*d*\vec{J}_b = 0$ is equivalent to $\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0$

But this follows directly from Maxwell's inhomogeneous equations

$$*d*\vec{J}_b = \frac{1}{\mu_0}*d*(*d*F) = -\frac{1}{\mu_0}*dd*F = 0$$

Note that defining $J^0 = \rho$ is a good choice because of continuity of charge.

Thank You Solomon Akaraka Owerre

Most of the information described here is extracted from a paper by
Solomon Akaraka Owerre.

Maxwell's Equations in Terms of Differential Forms
https://scholar.google.ca/citations?user=ZcQiAIMAAAAJ&hl=en
https://bbs.pku.edu.cn/attach/13/c8/13c819b28e8fb43c/maxwell_hodge.pdf

My only addition to his beautiful expression of this topic is the inclusion of ϵ_0, μ_0, and c.

Questions and Comments are Welcome jed@islandnet.com