16. Reference Frames in General Relativity

With the Einstein Equivalence Principle, we thought of falling inertial reference frames. The concept of a moving reference frame, only makes sense when there is a related observer coordinate system. In the case of a space time manifold, motions and events can appear very different from different coordinate systems. This is analogous to the difference in appearance of great circles in different geographic map projections.

An inertial coordinate system has its origin at a particular point in space and time. For our physical intuition, we use poetic license and think of it as stationary and falling. Expect different spatial and temporal effects in different coordinate systems.

Geodesics and Gravity

In the general theory of relativity, local mechanics is expressed in the stress-energy tensor. Body forces such as gravity are not expressed in terms of stress. Given that gravity effects everything the same way, we expect gravity to be a property of space-time.

Consider a puff of smoke initially spatially at rest in an inertial space-time reference frame, falling straight toward the center of the earth. The particle of smoke at the origin travels strait forward in time, but the space coordinates remain fixed. This particle travels along a particularly simple geodesic.

A smoke particle initially at rest away from the origin curves toward the spatial origin as it travels forward in time. This would not happen if space-time were flat.

The projections of the space-time geodesics onto 3-space converge toward the center of the earth (which is accelerating toward the space origin in our coordinate system). In this inertial coordinate system, each particle of smoke follows its own geodesic. We see this as tidal forces lengthening and squeezing the puff of smoke.

Gauss and Flat Space

Gauss gave us intrinsic curvature. There is a legend that Gauss attempted to experimentally determine if space was flat by measuring the angles in a large triangle of lines of sight from mountain tops. We will use this idea of experimental spatial geometry in our description of General Relativity.

The Stress-Energy Tensor in Non-Flat Space-Time

Lets do a thought experiment where we assemble a continuum of matter in motion and subject to stress, in an inertial (falling) reference frame. At any point, the ordinary Cartesian divergence of the stress-energy tensor should be zero, if space-time is flat.

If there are tidal forces on the matter, the divergence of the stress-energy tensor will not be zero. Maybe there is a pseudo-metric where the covariant divergence of the stress-energy tensor is zero. This expresses special relativity mechanics in the tangent plane.

So, with the Einstein Equivalence Principle in mind, and body force effects replaced by motion along geodesics, we get a relationship between the geometry of space and the stress-energy tensor. In our thought experiment, how would we find an appropriately related pseudo-metric?

The Einstein Tensor

The Einstein tensor denoted as G_{ij} , is calculated from any pseudo-metric and its derivatives, it expresses the confluence of geodesics.

It is defined by $G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R$

where

g_{ij} is the pseudo-metric

$R_{ij} \equiv R^{k}_{ikj}$ is the Ricci tensor

$R^{\alpha}_{i\beta j}$ is the Riemann tensor

$R \equiv g^{\alpha\beta} R_{\alpha\beta}$ is the Ricci scalar curvature (not Riemannian curvature)

The importance of G_{ij} here, is that the covariant divergence of G^{ij} is always zero. So, for a given arrangement of matter expressed by the stress-energy tensor T^{ij} , we can guarantee that its covariant divergence is zero, if we set $G_{ij} = T_{ij}$. This expresses a relationship between the pseudo-metric and the stress-energy tensor.

We can continue to guarantee the zero covariant divergence of the stress-energy tensor even if we introduce some constants into the above expression.

Einstein Field Equation

We can express the relationship between the stress-energy tensor and the pseudo-metric with more constants as

$G_{ij} + \Lambda g_{ij} = \kappa T_{ij}$ where Λ and κ are any constants we want .

So far this expresses the consistency of relativistic mechanics and geometry. We have a physical law when we choose the constants.

We require that the pseudo-metric be appropriate for a space-time manifold, so that it can be expressed as the Minkowski metric at any point.

The constant Λ is chosen for the requirements of cosmological modeling.

The constant κ is chosen so that we have consistency with Newtonian gravity for a small stationary mass in a small region of space-time. Note that all the components of the stress-energy tensor are related to the pseudo-metric. Newtonian theory only uses mass.

The Einstein Field Equation (EFE) can be expressed as

$$G_{ij}+\Lambda g_{ij}=\frac{8\pi G}{c^4}T_{ij}$$

where G is Newton's gravitational constant and c is the speed of light.

This is a good theory because:
Physics is consistent in the tangent plane (or in a small region of space-time).
Special relativity is included in the theory.

In addition to the theoretical consistency of the EFE, simplified cases for stress-energy tensors have been used to approximate astronomical configurations that suggested observations that have confirmed the EFE in contrast to Newtonian theory. The most famous observations are the perihelion of mercury and the bending of light passing a star.

Justifying the Einstein Tensor
Einstein found a usable tensor after four years of trial and error (possibly directed by the behavior of geodesics). He did this at a time when various ideas of differential geometry were under development (Einstein notation wasn't available). With this apparently artificial discovery, the fundamental nature of the Einstein tensor is surprising.

Just before Einstein published his tensor, it was derived by Hilbert via a least action principle, involving Ricci curvature. This was additional evidence to the fundamental nature of the Einstein tensor.

Much later, it was proved that the Einstein tensor is essentially the only tensor with zero covariant divergence that results from only a few derivatives of the pseudo-metric.

Gravity and Matter
The EFE and Newton's theory both describe what gravity does, but not how it does it. In any case, there seems to be something connecting gravity and matter, in the Newtonian limit.

In the Newtonian limit, the only significant element of the stress-energy tensor is matter, because of the high speed of light. So, if the non-zero tidal motion is due to geometry, only matter can effect the geometry.

Note that Newtonian theory equally emphasizes both bodies in a two-body system. In Einstein theory, the emphasis is divided between matter and motion creating geometrical effects on test particles with very small mass. The equivalence of inertial mass and gravitational mass is subtly included in this geometric view of gravity.

Gravity in Empty Space

Solving the EFE for a mechanical system is usually difficult. Simplified assumptions about the distribution of the mechanics lead to useful results. In Schwarzschild 's example, the stress-energy tensor is spherically symmetric, with stationary matter, and bounded within a sphere. In the empty space outside the sphere, the stress energy tensor and Einstein tensor must be zero. So it is easy to examine properties of the pseudo-metric in the empty space outside the sphere. Here, the Ricci curvature is zero, but not the Riemannian curvature.

This is similar in spirit to determining the Newtonian inverse square law in the empty space around a spherical body from $\nabla^2\Phi=0$.

Einstein Gravitation Summary

Before the General Theory of Relativity, it was not clear how to calculate relativistic effects in the presence of a gravitational field. The Einstein Equivalence principle gives direction on what the theory should be.

The line of thought for Einstein's theory of gravity goes like this:

> From EEP, we get that gravity is universal.
>
> Tidal effects are described as geodesic motion.
>
> Relativistic mechanics are described by the covariant derivative of the Stress-energy tensor.
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> Geometry and mechanics must be related for mechanical consistency.