15. Classical Stress Tensor

The stress tensor is used in continuum mechanics and figures prominently in Einstein's general theory of relativity. We use the classical stress tensor to calculate the surface force on a blob of stuff in terms of tension and shear forces. Here, we ignore body forces such as gravity. We use a force-vector valued integral expression. $\vec{f} = \oiint_{\text{surface}} d\vec{A}^t \cdot S$.

Here, the stress tensor $S^{\alpha\beta}$ is a 3x3 matrix where each row represents the force density vector between the corresponding coordinate planes. The diagonal elements represent tension. The off-diagonal elements represent shear. A coordinate sign convention is also involved.

$$S^{\alpha\beta} = \begin{bmatrix} S^{xx} & S^{xy} & S^{xz} \\ S^{yx} & S^{yy} & S^{yz} \\ S^{zx} & S^{zy} & S^{zz} \end{bmatrix}$$

Each column represents a vector field of flux contributing to the corresponding component of force. Each component of net force on the blob is the volume integral of the divergence of the corresponding column of $S^{\alpha\beta}$.

Using the stress tensor $S^{\alpha\beta}$ to express internal forces within a body, we find that $S^{\alpha\beta}$ is symmetric. This is still true even when there is a net torque on the blob. This symmetry reflects a restriction in our model that, blobs within blobs do not tend to roll over each other, or the fluid is not turbulent.

Classical Stress-Momentum Tensor

We will expand the shape of the stress tensor so that it includes momentum and density. We recognize that force is the time derivative of momentum at a point where the matter is at rest.

We express this as $\frac{\partial p^j}{\partial t} = \frac{\partial S^{xj}}{\partial x} + \frac{\partial S^{yj}}{\partial y} + \frac{\partial S^{zj}}{\partial z}$

We write the momentum as the zeroth row of the matrix.

$$T^{\alpha\beta} = \begin{bmatrix} -p^x & -p^y & -p^z \\ S^{xx} & S^{xy} & S^{xz} \\ S^{yx} & S^{yy} & S^{yz} \\ S^{zx} & S^{zy} & S^{zz} \end{bmatrix}$$

The previous expression is useful because the divergence of each column is zero.

$$0=-\frac{\partial p^j}{\partial t}+\frac{\partial S^{xj}}{\partial x}+\frac{\partial S^{yj}}{\partial y}+\frac{\partial S^{zj}}{\partial z}$$

We can call the top row, the temporal row. Also, this is an expression of f=ma.

We also note that conservation of mass (continuity) can be expressed as

$$-\frac{\partial \rho}{\partial t}=\frac{\partial p^x}{\partial x}+\frac{\partial p^y}{\partial y}+\frac{\partial p^z}{\partial z}$$ (momentum is matter flux)

So, we can create a zeroth (temporal) column and write

$$T^{\alpha\beta}=\begin{bmatrix}-\rho & -p^x & -p^y & -p^z \\ -p^x & S^{xx} & S^{xy} & S^{xz} \\ -p^y & S^{yx} & S^{yy} & S^{yz} \\ -p^z & S^{zx} & S^{zy} & S^{zz}\end{bmatrix}$$

The divergence of every column is zero and the matrix is symmetric.

Without discussing the fluid mechanics details, we can adjust the spatial part of the matrix (with products of velocity terms) to account for matter in motion. This allows for cases like $\vec{p}=0$ but the material is accelerating. We still have zero divergence.

$$T^{\alpha\beta}=\begin{bmatrix}-\rho & -p^x & -p^y & -p^z \\ -p^x & T^{xx} & T^{xy} & T^{xz} \\ -p^y & T^{yx} & T^{yy} & T^{yz} \\ -p^z & T^{zx} & T^{zy} & T^{zz}\end{bmatrix}$$

Pseudo Riemannian Manifold

A pseudo Riemannian manifold is like a Riemannian manifold except that the metric is not positive definite. In this case we have to live with some tangent vectors having a negative length and two distinct tangent vectors being zero distance apart. This means that more poetic license is required when using geometric visualization and diagrams.

Pseudo Riemannian manifolds appear in physics when the time variable is grouped with the space variables. This happens indirectly with Newtonian Lagrangian mechanics, and more directly with electrodynamics and special relativity.

Minkowski Pseudo-Metric

The Minkowski pseudo-metric on 4-d space-time is used instead of the Euclidean metric because it provides a unifying approach to electrodynamics and special relativity. One way of expressing it is:

$$g_{ij}=\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The spatial part is represented by the lower right 3x3 matrix. Temporal components of tangent vectors are affected by the zeroth row and column. This pseudo-metric is significant because it is invariant under Lorentz transformations.

Stress-Energy Tensor

We have seen that the classical stress-momentum tensor for a continuous Newtonian system without body forces, has a zero divergence. The classical stress-momentum tensor, can be corrected to represent the mechanics of special relativity. This is done by embedding factors of $\sqrt{1-v^2/c^2}$ and c where required in the components of the tensor. The relativistic stress-energy tensor can be expressed as

$$T^{\alpha\beta}=\begin{bmatrix} -E/c^2 & -p^x & -p^y & -p^z \\ -p^x & T^{xx} & T^{xy} & T^{xz} \\ -p^y & T^{yx} & T^{yy} & T^{yz} \\ -p^z & T^{zx} & T^{zy} & T^{zz} \end{bmatrix}$$

We have omitted discussing issues involving the tensorial nature of the divergence and upper and lower indices.

Conveniently, local coordinate systems in space-time can be chosen (by tilting the time axis) so that any particle is momentarily at rest. This simplifies the stress-energy tensor at the origin.

Stating that the divergence of the stress-energy tensor is zero is a way of expressing the mechanics of special relativity.

Space-time Manifolds

A 4-d manifold that has a pseudo-metric that can be transformed into the Minkowski pseudo-metric at any point is a sufficiently general model that expresses local special relativity without reference to a special origin.

A space-time manifold provides the framework to express the General Theory of Relativity. This theory expresses the extension of special relativity to environments that are subject to gravity.

Moving Reference Frames in Space-Time
In Newtonian 3-space, we think of a moving reference frame in the context of a path indexed by time, which is not one of the Newtonian spatial coordinate variables. In space-time we would expect to use another path parameter such as τ , which might not mean time.

In special relativity, we might express the path of a moving reference frame in terms of a special observer coordinate system. Note that for each value of τ there is a single unambiguous origin for the origin of the moving reference frame. It is usually convenient (and confusing) to set the path parameter τ to the time coordinate of the observer. Special relativity deals with the relationship between the space-time coordinates of these two coordinate systems.

Newtonian Gravity
Inverse square force fields have a pleasing artistic representation as lines of force and it is difficult to think of 3-d force fields in any other way than in the context of Gauss' law geometry. The inverse square law of gravity is geometrically elegant, mathematically tractable and has survived scrutiny for two centuries.

Local effects of matter in Newtonian gravitation can be expressed in terms of the gravitational potential function as $\nabla^2 \Phi = -4\pi G \rho$, where G is Newton's gravitational constant. This differential equation in the context of a boundary-value problem gives a global specification of the gravitational field, even away from the matter. This spirit of localness will be used in Einstein's theory of gravity.

Newtonian gravitational theory shows its weakness in the context of relativity effects such as mass energy equivalence and the gravitational red shift.

The Einstein Equivalence Principle
A basic idea in Newtonian mechanics is that, physics behaves the same way in any reference frame traveling at a constant velocity. The basic form of the Einstein Equivalence Principle (EEP) is a similar idea. Here, physics behaves the same way in any falling reference frame. This EEP is not quite true because of the presence of gravitational tidal forces. However, the EEP is true in the limit of small regions of space-time.

The EEP is a universality concept directing us to believe that gravity effects everything, including light and electromagnetic fields, not just matter.