14. General Laplacean

The Laplacean is a second derivative expression that frequently occurs in physics. In 2-d it can be expressed as $\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ a scalar field. In vector calculus, it is described as the divergence of the gradient. Because of its wide use in physics, we hope to have a tensor description that can pass through changes of coordinates. In theory, all we need to do is follow the chain rule through all the derivatives. This can be very complicated. We have a better way.

Physics and the Laplacean

The Laplacean occurs frequently in the studies of fields, heat conduction, and waves. We illustrate wave examples of the Laplacean. We first recall the 1-d analysis of a wavy rope and its displacement function f. We consider the rope density ρ and its tension T. After using the small displacement approximation we get the 1-d wave equation for the displacement of a wavy rope:
$\frac{\partial^2 f}{\partial x^2} = \sqrt{\frac{\rho}{T}} \frac{\partial^2 f}{\partial t^2}$. It relates rope curvature and acceleration. It has sinusoidal solutions like $f(x, t) = \sin(kx - \omega t)$,where $\frac{\omega}{k} = \frac{T}{\rho}$

For a 3-d example we can examine the wave equation for sound pressure waves $P(x, y, z; t)$ (pressure difference from ambient pressure). We can do a physical analysis on fluid density ρ and elasticity B. After using suitable approximations, we get the wave equation:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = \nabla^2 P = \sqrt{\frac{\rho}{B}} \frac{\partial^2 P}{\partial t^2}$$

A physicist can stare at this equation and see that there are plane wave solutions. When we change to spherical coordinates we can examine the new expression (a little complicated) and see that there are spherical wave solutions. In many applications it is useful to change coordinates to fit the geometry of a boundary.

Vector Fields and Differential Forms

Recall our discussion of the divergence theorem and Green's theorem, in the context of the General Stokes theorem. In the case of the traditional Green's theorem, we viewed a vector field as a 1-form. In the case of the traditional Divergence theorem, we viewed a vector field as a 2-form. We examine this more closely on the way to the general Laplacean.

Inner Product, Sharp, and Flat

We define inner product with angle bracket notation as
$\langle \vec{x}, \vec{y} \rangle \equiv x^i g_{ij} y^j$ on vectors and $\langle \tilde{p}, \tilde{q} \rangle \equiv p_i g^{ij} q_j$ on covectors

Given a vector $\vec{x} = x^i$, we can convert it to a covector $\vec{x}_b \equiv x^i g_{ij} \equiv x_j$
so that $\vec{x}_b(\vec{y}) = \langle \vec{x}, \vec{y} \rangle$. This is also called lowering an index on x^i to x_j . And we borrow a term from a musically lower note and call this operation flattening.

Given a covector $\tilde{q} = q_i$, we can convert it to a vector $\tilde{q}^\# \equiv q_i g^{ij} \equiv q^j$
so that $\tilde{p}(\tilde{q}^\#) = \langle \tilde{p}, \tilde{q} \rangle$. This is also called raising an index or sharpening.

Then $\langle \tilde{p}, \tilde{q} \rangle = \langle \tilde{p}^\#, \tilde{q}^\# \rangle$ and $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}_b, \vec{y}_b \rangle$

In Euclidean space, we don't see the effect of the metric when we raise and lower indices. This makes it harder to distinguish between row and column vectors.

It is traditional to express the gradient of a scalar valued function as a vector by raising its covector of partial derivatives $\nabla f = g^{ij} \dfrac{\partial f}{\partial x^i}$

A better statement of the traditional Green's theorem for a vector field $\vec{F}$ is

$$\int_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx \wedge dy = \int_A d\vec{F}_b = \int_{\partial A} \vec{F}_b = \int_{\partial A} \left(F_x dx + F_y dy \right)$$

The inner equality is valid in all coordinate systems.

Hodge Star

We introduce the Hodge Star operation on k-forms with some examples from 3-d Cartesian space.

$$*dx = dy \wedge dz \quad , \quad *(dx \wedge dy) = dz \quad , \quad *1 = dx \wedge dy \wedge dz \quad , \quad *(dx \wedge dy \wedge dz) = 1$$

A better statement of the traditional divergence theorem for a vector field $\vec{F}$ is

$$\int_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx \wedge dy \wedge dz = \int_V d\left(*\vec{F}_b \right) = \int_{\partial V} *\vec{F}_b$$

$$= \int_{\partial V} \left(F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy \right)$$

The Hodge star of a k-form in Euclidean space is the (n-k)-form that combines with the k-form to express the determinant via the wedge product. It is alternatively called the Hodge dual. Note that it does not create an element in the algebraic dual space of k-forms. However, it does map elements to the algebraic dual in the context of homology groups. We will not be going there.

Most people are using the Hodge Star when doing flux integration through a surface in 3-d Cartesian space. A 2-form acting on surface elements becomes a 1-form acting on surface normals. The 1-form is represented as a vector field projected onto surface normals.

To make the Hodge star concept linear and tensorial (for orientation preserving transformations), it is defined so that:

$$\omega \wedge * \omega = \langle \omega , \omega \rangle \sqrt{|g_{ij}|}\ \det()$$

Here the inner product on k-forms is an extension of the inner product on 1-forms so that $\langle dx^1 \wedge \cdots dx^k\ ,\ dy^1 \wedge \cdots dy^k \rangle = \det\left(\langle dx^i , dy^j \rangle \right)$. In the case of the basic n-form $\langle \det\ ,\ \det \rangle = \langle dx^1 \wedge \cdots dx^n , dx^1 \wedge \cdots dy^n \rangle = \det\left(g^{ij}\right) = \dfrac{1}{\det\left(g_{ij}\right)}$.

use $\det \wedge \dfrac{1}{\sqrt{g_{ij}}} = \dfrac{1}{\det\left(g_{ij}\right)} \sqrt{|g_{ij}|}\ \det() = \langle \det\ ,\ \det \rangle \sqrt{|g_{ij}|}\ \det$ to get $* \det() = \dfrac{1}{\sqrt{|g_{ij}|}}$

Note the antisymmetry effect in 2-d Cartesian space $*dx = dy \qquad *dy = -dx$

Expressing the General Laplacean

Suppose we did a physical analysis and arrived at the Laplacean differential expression. We would like to express it as a tensor that will be valid in all coordinate systems. This works: $\nabla^2 f = * d * d\ f$. We see that this is a second derivative expression. The derivatives are tensorial because they are exterior derivatives. The right-most derivative creates a 1-form. The rightmost * converts it to an (n-1)-form. The left-most derivative converts it to an n-form. The left-most * converts it to a 0-form or scalar field. This scalar field will be the same in any coordinate system. There are also other related expressions for the Laplacean. We choose this one because of its clear relationship to stokes theorem.

For 2-d Cartesian space

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \qquad\qquad *df = \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx$$

$$d * df = \frac{\partial^2 f}{\partial x^2} dx \wedge dy + \frac{\partial^2 f}{\partial y^2} dx \wedge dy \qquad\qquad * d * df = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Calculating the Laplacean

Before calculating the Laplacean in other coordinate systems, we need g_{ij} and g^{ij}. To calculate the Hodge stars we need to calculate

$$\langle df\ ,\ df\rangle \quad \text{and} \quad \langle d{*}df\ ,\ d{*}df\rangle\ .$$

These inner products are easy to calculate because they are either 1-forms or n-forms.

Polar Coordinate Laplacean

For polar coordinates $(r\ ,\ \theta)$ we have the Riemannian metric and its inverse:

$$g_{ij}=\begin{bmatrix}1 & 0\\ 0 & r^2\end{bmatrix} \qquad g^{ij}=\begin{bmatrix}1 & 0\\ 0 & \dfrac{1}{r^2}\end{bmatrix}$$

Use $dr\wedge{*}dr=\langle dr\,,dr\rangle\sqrt{|g_{ij}|}\,\det(\,)=(1)(r)(dr\wedge d\theta)$ to get ${*}dr=r\,d\theta$

Use $d\theta\wedge{*}d\theta=\langle d\theta\,,d\theta\rangle\sqrt{|g_{ij}|}\,\det(\,)=-\left(\dfrac{1}{r^2}\right)(r)(d\theta\wedge dr)$ to get ${*}d\theta=-\dfrac{1}{r}dr$

Also ${*}(dr\wedge d\theta)={*}\det(\,)=\dfrac{1}{\sqrt{|g_{ij}|}}=\dfrac{1}{r}$

To calculate $\nabla^2 f={*}d{*}d\,f$

Begin with $df=\dfrac{\partial f}{\partial r}dr+\dfrac{\partial f}{\partial\theta}d\theta \qquad {*}df=r\dfrac{\partial f}{\partial r}d\theta-\dfrac{1}{r}\dfrac{\partial f}{\partial\theta}dr$

Then $d{*}df=\dfrac{\partial f}{\partial r}dr\wedge d\theta+r\dfrac{\partial^2 f}{\partial r^2}dr\wedge d\theta+\dfrac{1}{r}\dfrac{\partial^2 f}{\partial\theta^2}dr\wedge d\theta$

Finally $\nabla^2 f={*}d{*}df=\dfrac{1}{r}\dfrac{\partial f}{\partial r}+\dfrac{\partial^2 f}{\partial r^2}+\dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial\theta^2}=\dfrac{\partial^2 f}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial f}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial\theta^2}$