11. Integrating Around a Rectangle

Before we work with higher order forms, we illustrate the relationship between boundary integration and the derivative inside the boundary. In this elementary example we integrate a vector field f(x,y) around a rectangle, as a line integral. The rectangular interior can be expressed as $x_1 \le x \le x_2$ and $y_1 \le y \le y_2$. The bottom and top edges can be indicated by $+\Delta \vec{x}$ and, $-\Delta \vec{x}$ with the right and left edges indicated by $+\Delta \vec{y}$ and, $-\Delta \vec{y}$.

Then the integral around the boundary can be approximated (with loose notation) as:

$$\oint f(x,y) \cdot ds \approx f_x(\bar{x}, y_1)\, \Delta x + f_y(x_{2,} \bar{y})\, \Delta y - f_x(\bar{x}, y_2)\, \Delta x - f_y(x_{1,} \bar{y})\, \Delta y$$

We regroup the terms as pairs of line integrals on opposite sides of the rectangle.

$$\oint f(x,y) \cdot ds \approx \left(f_x(\bar{x}, y_1) - f_x(\bar{x}, y_2)\right) \Delta x + \left(f_y(x_{2,} \bar{y}) - f_y(x_{1,} \bar{y})\right) \Delta y$$

Next we estimate the difference between the pair elements with the corresponding derivatives across the rectangle.

$$\oint f(x,y) \cdot ds \approx -\frac{\partial f_x(\bar{x}, \bar{y})}{\partial y}\, \Delta x \Delta y \; + \; \frac{\partial f_y(\bar{x}, \bar{y})}{\partial x}\, \Delta x \Delta y = \left(-\frac{\partial f_x(\bar{x}, \bar{y})}{\partial y} + \frac{\partial f_y(\bar{x}, \bar{y})}{\partial x}\right) \Delta x \Delta y$$

This is an approximation to an integral over the interior. This idea is a hint of the relationship between the integral over a boundary and an integral of a derivative expression on the interior. This story continues when we consider the exterior derivative of a differential form. We start with a 1-form.

Differentiating a 1-Form

In 3-d space, consider a 1-form given by $\omega = f\, dy$ and a 2-form given by

$$\begin{aligned} d\omega &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right) \wedge dy \\ &= \frac{\partial f}{\partial x} dx \wedge dy + \frac{\partial f}{\partial y} dy \wedge dy + \frac{\partial f}{\partial z} dz \wedge dy \\ &= \frac{\partial f}{\partial x} dx \wedge dy - \frac{\partial f}{\partial z} dy \wedge dz \end{aligned}$$

Lets integrate $d\omega$ over the unit 2-cell C in the xy-plane. Only the $dx \wedge dy$ term contributes to the integral:

$$\int_C d\omega = \int_0^1 \int_0^1 \frac{\partial f}{\partial x} dx\, dy = \int_0^1 \left(f(1,y,0) - f(0,y,0)\right) dy$$

$$=\int_{(1,0,0)}^{(1,1,0)} f\,dy-\int_{(0,0,0)}^{(0,1,0)} f\,dy=\int_{\partial C}\omega$$

Note that top and bottom edges do not contribute to the boundary integral.

This demonstrates the fundamental theorem of calculus applied to 1-forms where the change in value of the 1-form is related to the integral of a 2-form.

If we integrate over the unit 2-cell in the yz-plane, a minus sign has been introduced so that

$$\int_C d\omega=-\int_0^1\int_0^1 \frac{\partial f}{\partial z}\,dy\,dz=-\int_{(0,0,1)}^{(0,1,1)} f\,dy+\int_{(0,0,0)}^{(0,1,0)} f\,dy=\int_{\partial C}\omega$$

and our expression is compatible with assignment of signs to boundary elements.

In general $\int_C d\omega=\int_{\partial C}\omega$ For any 2-cell.

Exterior Derivatives
We have seen the construction of a 1-form through differentiation of a 0-form.

In general, we build the exterior derivative of a basic k-form term $\omega=f\wedge\left(dx^{\sigma_1}\wedge\cdots dx^{\sigma_k}\right)$, by summing terms over all partial derivatives:

$$d\omega=\left(\sum_j \frac{\partial f}{\partial x^j}dx^j\right)\wedge\left(dx^{\sigma_1}\wedge\cdots dx^{\sigma_k}\right)$$

However, we exclude degenerate terms by applying the rule $dx^i\wedge dx^i=0$.

For example, in the case of xyz space and the 2-form $\omega=f\,dy\wedge dz$, the exterior derivative is given by the single term $d\omega=\frac{\partial f}{\partial x}dx\wedge dy\wedge dz$

For the basic k-form ω, the support of $d\omega$ is characterized by all (k+1)-cells with one face in the the subspace spanned by $\hat{x}_{\sigma_1}\cdots\hat{x}_{\sigma_k}$.

Coordinate Free Exterior Derivative of a 1-Form
This special case will be useful when we study curvature. Consider a 1-form ω acting on arbitrary vector fields. An expression for the exterior derivative $d\omega$ acting on a pair of vectors can be shown by brute calculation to be:

$$d\omega(\vec{V},\vec{W})=\nabla_{\vec{V}}\,\omega(\vec{W})-\nabla_{\vec{W}}\,\omega(\vec{V})-\omega\left(\left[\vec{V},\vec{W}\right]\right)$$

The directional derivatives are applied to the scalar valued functions defined by ω.

The Lie bracket $\left[\vec{V},\vec{W}\right]\equiv\nabla_{\vec{V}}\vec{W}-\nabla_{\vec{W}}\vec{V}$ signifies the difference in directional derivatives. Normally, the directional derivative of a vector field is not a tensor. In this case, the expression is a tensor because certain terms cancel. This is also a demonstration that the exterior derivative of a 1-form is a tensor field, and this extends to the other k-forms.

Applying the Exterior Derivative Twice

We recall an identity from vector calculus involving the curl of the gradient. Later, we will demonstrate that the gradient and curl are examples of the exterior derivative.

$$\nabla\times(\nabla f)=\nabla\times\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y\partial x}-\frac{\partial^2 f}{\partial x\partial y}=0$$

This example demonstrates the combination of two derivatives that yield zero.

There is a similar property for the exterior derivatives of k-forms: $dd\omega=0$. We will see that $d\omega$ is defined so that it is compatible with the fundamental theorem of calculus on all (k+1)-cells orthogonal to the k-dimensional support of ω . So, $d\omega$ has terms for all the independent relevant (k+1)-cells.

When we apply the exterior derivative operation a second time, we wind up visiting terms with second derivatives twice, with opposing signs, and thereby canceling to zero.

We illustrate this with the 1-form $\omega=f\,dz$ in xyz-space.

$$d\omega=\frac{\partial f}{\partial x}dx\wedge dz+\frac{\partial f}{\partial y}dy\wedge dz$$

$$dd\omega=\frac{\partial^2 f}{\partial x\partial y}dy\wedge dx\wedge dz+\frac{\partial^2 f}{\partial y\partial x}dx\wedge dy\wedge dz=0$$

.

dd=0 and $\partial\partial=0$

There is a notable similarity between properties of exterior derivatives of forms and boundary operations on cells. If we followed this line of thought, we would study algebraic topology and arrive at De Rham's theorem.

General Stokes' Theorem on a Unit Cell

We have had examples of $\int_C d\omega=\int_{\partial C}\omega$ for 0-forms and basic 1-forms. We now consider a basic k-form $\omega=f\,dx^{\sigma_1}\wedge\cdots dx^{\sigma_k}$ applied to opposing faces of a (k+1)-cell C aligned with the coordinate axes. The pair of faces are identified from $dx^{\sigma_1}\wedge\cdots dx^{\sigma_k}$ and one more orthogonal coordinate direction $\hat{x}_j$. We find that the only contribution to $\int_{\partial C} \omega$ is from the pair of opposing faces, at opposite ends of $\hat{x}_j$ with opposite signs. As with the previous examples, We get $\int_C d\omega = \int_{\partial C} \omega$. The order of subtraction along the boundary is compatible with the even or odd permutation of $dx^{\sigma_1} \wedge \cdots dx^{\sigma_k}$. The details aren't shown here.

General Stokes' Theorem on General Regions

We have demonstrated the General Stokes Theorem $\int_C d\omega = \int_{\partial C} \omega$ For carefully selected cells. But the relation is also true for any differentiable image of a unit cell because we have been dealing with tensors which have the same properties as we change between charts. An important consideration here is that wedge products and $\frac{\partial f}{\partial x^1} dx^1 \wedge \cdots$ transform properly, which in turn makes the exterior derivative of a k-form also a tensor field.

We can push the generality further to stacking images of cells because interior boundaries cancel. And finally, the theorem applies to regions that can be approximated by stacks of cell images.

Green's Theorem Revisited

We can write the line and surface integrals involved with the traditional Green's theorem in terms of differential forms.

For example, consider integrating around a loop bounding a region A, in the xy-plane like $\oint \vec{F} \cdot d\vec{s}$. We express this using a 1-form $\omega = F_x dx + F_y dy$ acting on the 1-cell images that express the boundary.

After a bit of algebra, we have $d\omega = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx \wedge dy$

So $\int_A d\omega = \int_{\partial A} \omega \quad \Rightarrow \quad \int_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx \wedge dy = \int_{\partial A} F_x dx + F_y dy$

which has the appearance of the traditional Green's theorem.