10. Parallel Transport

The idea of parallel transport means moving a vector along a path without changing its length or direction. This is described mathematically as a vector field, in a neighborhood of the path that has a zero covariant derivative along the path.

Consider our example of the arctic circle that flattens to an arc. We see that a vector field that is initially pointing east at the west end of the arc, must be rotated south at the east end, if its covariant derivative will remain zero. Parallel transport is a concept that can be used to study intrinsic curvature.

Differential Forms and Boundaries

We will now look at the relationship between the derivative of a differential form in the interior of a region and its value on the boundary, These ideas are commonly used in Green's theorem and the divergence theorem. We will show that these theorems are special cases of a more general theorem involving differential forms.

Our technique is to look at basic differential forms on the simple geometry of squares and cubes. The linearity of differential forms and the flexibility of changing coordinate systems allows us to generalize results to more general regions.

Eventually, we will explore differentiation and boundaries of regions to understand the meaning of the general Stokes' theorem:

$$\int_C d\omega = \int_{\partial C} \omega$$

We now review material related to traditional Divergence and Green's Theorems.

Divergence and Curl

This is a review of two important derivative operations from vector calculus. They are frequently used in the study of electromagnetic fields and fluids. We will see these differential operations later as special cases of derivatives of differential forms.

In 3-d space, consider a vector field given by $\vec{F} = F_x \hat{x} + F_y \hat{y} + F_x \hat{z}$ Then the divergence is a scalar field defined by $\operatorname{div} \vec{F} \equiv \nabla \cdot \vec{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$. It describes the rate of change of $\vec{F}$ in the direction of flow. For example, if $\vec{F}(x, y, z) = (a x + 1)\hat{x}$, then $\nabla \cdot \vec{F} = a$. It is typically used to describe non-compressible fluid flow where $\nabla \cdot \vec{F} = 0$.

The curl is a vector field defined by

$$\text{curl } \vec{F} \equiv \nabla \times \vec{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$$

It describes the rate of change of $\vec{F}$ across the direction of flow. It is typically used to describe fluid rotation or shear. The vector points along the axis of rotation (or orthogonal to the shear). A shear example is given by $\vec{F}(x, y, z) = (a x + 1)\, \hat{y}$ and $\nabla \times \vec{F} = a \hat{z}$.

Later, we will see that these derivatives are special cases of the exterior derivative of differential forms. Furthermore, these derivative operations are intimately related to corresponding integral operations and boundary considerations. They express an extension of the fundamental theorem of calculus.

Green's Theorem

You may recall learning about the relation between the curl of a vector field in a surface and the corresponding line integral around is boundary:

. $\iint \nabla \times \vec{F} \cdot d\vec{A} = \oint \vec{F} \cdot d\vec{s}$

This theorem is typically used to relate the flow around a boundary to the net rotation in its interior. Consider our previous shear example $\vec{F}(x, y, z) = (a x + 1)\, \hat{y}$ in the unit xy-square. This is a vector field pointing in the $\hat{y}$ direction with increasing magnitude as we move in the $\hat{x}$ direction. There is no contribution to the line integral on the y=0 and y=1 edges. The line integral is essentially the difference of $\vec{F}$ between the left and right edges. This is an example where the integral of a derivative of a function is obtained from the difference of its value on the boundary.

Divergence Theorem

In your vector calculus course you may remember flux integrals and their relationship to the divergence of a vector field:

$$\iiint \nabla \cdot \vec{F}\, dx\, dy\, dz = \oiint \vec{F} \cdot d\vec{A}$$

For example, consider our previous example $\vec{F}(x, y) = (a x + 1)\, \hat{x}$ on the unit cube. The only contribution to the flux integral is on the two faces perpendicular to the $\hat{x}$ direction. The total flux integral is essentially the difference of $\vec{F}$ between the left and right faces. Again, we see how the integral of a derivative of a function is obtained from the difference of its value on the boundary.

Cells
We will examine the geometry of regions and their boundaries through the simple geometry of cells. Examples of the simplest cells are the unit cube, unit square, and unit interval. A single point is a 0-cell. Faces of (k+1)-cells are k-cells. More general geometry can be studied by stacking contiguous differentiable images of cells to create more general shapes.

Parametrized Line and Surface integrals
Differentiable curves, surfaces and k-surfaces ultimately are locally parametrized by a k-cell. We can view a k-surface as a manifold within a manifold where a k-cell is its chart. Any k-form in the manifold can be represented as a k-form on the k-cell.

For example, a curve can be expressed as $(x(t), y(t), z(t))$ for $a \le t \le b$ so that it is an image of the 1-cell [a,b]. Then a 1-form like f dx + g dy + h dz is expressed like $\left(f\frac{dx}{dt}+g\frac{dy}{dt}+h\frac{dz}{dt}\right)dt$ in [a,b].
We will see this again when we revisit the Divergence and Green's Theorems.

Boundaries and Integration
Consider a 1d integral on an interval and the fundamental theorem of calculus. :

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a)$$

By assigning signs to the boundary points, we can write this as:

$$\int_{[a,b]} \frac{df}{dx} = \int_{\text{boundary}[a,b]} f$$

Furthermore, the signs of superposed boundary points is related to the cancellation of interior boundary points during integration:

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Cells and Boundaries
There is a combinatorial algorithm for assigning signs to boundary elements of k-cells so that interior boundaries algebraically cancel when the cells are packed together (with shared faces). This means that the assignment of boundary signs is compatible with oriented integration. For an assembly of cells C, we denote its signed boundary by ∂C .

For a 3-cell, opposite faces have opposite signs. A shared edge between two faces, is represented identically for each face, but with opposite sign. The collection of 2-cells that form the boundary of a 3-cell has a zero boundary. The combinatorial algorithm is designed to continue this pattern for all k-cells. This results in $\partial\partial C=0$ for any assembly of cells C. This is consistent with the idea that the boundary of a solid sphere (spherical shell) has no boundary.

To explain the combinatorial algorithm further, note that all k-faces of standard (k+1)-cells either have a corner at the origin or a corner antipodal to the origin (not both). The j^{th} k-face touching the origin is spanned by the k+1 standard basis vectors (in their natural order starting with 1) with the j^{th} basis vector removed. The face will get a positive sign if j is even. Iterate the boundaries of the j^{th} k-face by setting the natural order of spanning vectors from the removal of the j^{th} spanning vector from the parent spanning set.

1-Forms and the Fundamental Theorem of Calculus

Consider a 0-form $\omega=f(x,y,z)$ and its associated 1-form constructed from the gradient: $d\omega=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$

where $d\omega$ symbolizes the exterior derivative of ω (more later).

For an image of a 1-cell c from $\vec{a}$ to $\vec{b}$, we have the path integral:

$$\int_C d\omega=\int_{\vec{a}}^{\vec{b}}\frac{\partial f}{\partial x}dx+\int_{\vec{a}}^{\vec{b}}\frac{\partial f}{\partial y}dy+\int_{\vec{a}}^{\vec{b}}\frac{\partial f}{\partial z}dz=f(\vec{b})-f(\vec{a})=\int_{\partial C}\omega$$

This is a simple case of $\int_C d\omega=\int_{\partial C}\omega$

This case becomes even simpler if the 1-cell is aligned with one of the coordinate axes like $\hat{y}$. Then:

$$\int_C d\omega=\int_{\vec{a}}^{\vec{b}}\frac{\partial f}{\partial y}dy=f(\vec{b})-f(\vec{a})=\int_{\partial C}\omega$$

This case is as general as the first case because of the tensor nature of our quantities and the possibility of transforming to other charts.