9. Differential Forms Summary

We have presented differential forms as multi linear functionals for calculating the oriented area spanned by vectors. Because of their linear nature, they are tensors. Their relationship with determinants makes them skew symmetric. They are often used to express linear relationships between quantities that are approximately linear at small scales. They are frequently seen as differential expressions for calculating infinitesimal area or volume spanned by infinitesimal vectors.

Later, we will introduce the exterior derivative of differential forms. This will allow us to develop the general Stokes theorem, which extends the fundamental theorem of calculus.

A practical result of developing the mathematics of differential forms is that we have a mechanical way to change variables for integration in very general geometric settings. Without these methods, we use geometrical insight, such as spherical shells in radially symmetric applications.

Differentiating Vector Fields

For a scalar valued function on a manifold, the gradient provides us with a tensorial co-vector. Things are not so simple for vector fields and other linear objects. Complications arise because linear objects (such as vectors) are influenced by the changing tangent planes. If we restrict ourselves to a Cartesian space there is no difficulty.

Consider a vector field $\vec{v}$ in 2-d Cartesian space. Its directional derivative in the direction $\vec{w}$ is given in matrix and Einstein notation by:

$$\nabla_{\vec{w}}\vec{v} = \begin{bmatrix} \frac{\partial v^1}{\partial x^1} & \frac{\partial v^1}{\partial x^2} \\ \frac{\partial v^2}{\partial x^1} & \frac{\partial v^2}{\partial x^2} \end{bmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \frac{\partial v^i}{\partial x^j} w^j \quad \text{a vector.}$$

We will develop the idea of covariant derivative of a vector field on a manifold. The covariant derivative of a vector field is what you would expect if the manifold were Cartesian in the direction of differentiation. Then, we will discuss what this means mathematically.

Rolling a Plane along a Geodesic

If a manifold is embedded in a Cartesian space, a vector field on the manifold can be analyzed as though it is in the embedding space. For example, consider an east pointing unit vector field on the earth. Its derivative in the east direction is toward the earth axis. In this example, the derivative does not even lie in the manifold.

Sometimes we want a different type of derivative of the vector field. Imagine sailing a ship from Greenland to Norway straight east around the arctic circle. On the Atlantic chart it looks as though you should set your rudder straight when the rudder should be set to make a gentle turn to the north. We want to define a derivative of a vector field that describes its change within the surface and not depend on the embedding. We can visualize this idea in 2-d by rolling a plane across the surface without slipping or twisting.

Consider a vector field on a 2-d manifold and a geodesic in the direction of differentiation. Visualize a euclidean plane rolling (tipping) along the geodesic. During the roll, the local Cartesian coordinate systems are mapped into the tangent plane at each point of contact.

This match between the surface and the rolling plane is expressed mathematically by creating a geodesic coordinate system, described next.

Geodesic Normal Coordinates

At any point on a Riemannian manifold, we can construct a chart where the g_{ij} metric is euclidean at the point. This is done by choosing orthonormal members (as determined by the original g_ij) from the spray of geodesics emanating from the point. These choices correspond to the axes in the new chart. Grid lines in the new chart correspond to geodesics orthogonal to the initially chosen axis geodesics.

In more than two dimensions, the grid lines are recursively defined from geodesic surfaces orthogonal to the chosen axes.

It is plausible that the Riemannian metric expressed in the new chart is euclidean along the chart axes.

North Pole Example

Consider the North Pole. The lines of longitude are geodesics. We can choose the horizontal axes as the Greenwich meridian and 90° west longitude (through Siberia) as the vertical axes. Grid lines correspond to great circles tangent to circles of latitude at axis crossings.

In addition to the obvious discontinuity at the south pole, there is degeneracy at the equator. The chart is almost euclidean near either axis.

Flattening and Straitening along a Path

Along a path (not intersecting itself) we can create a continuum of geodesic coordinate systems with axes that are all correlated.

Consider rolling a plane around the arctic circle. In this case, it is easier to think of a cone tangent to the earth at the arctic circle. If the cone is cut and flatted, we see the arctic circle as a curved arc on a flat space. Notice that we had to remove a point from the arctic circle.

In the case of polar coordinates, the Cartesian coordinate system is a geodesic coordinate system that straightens the geodesics that look like arc cosine curves. We can get a flat image of any path by using geodesic coordinates. This is the mathematical realization of the rolling plane vision. This results in a chart that has the euclidean metric along the path.

Covariant Derivative

The covariant derivative of a vector field $\vec{v}$ at a point in the direction $\vec{w}$ is the Cartesian derivative in the chart for a coordinate system where the metric is euclidean in the $\vec{w}$ direction. So, the covariant derivative is a tensor because it is defined in a preferred coordinate system and then expressed in any other coordinate system via transition rules.

We will develop the expression $\delta_{\vec{w}}\vec{v} \equiv \left(\frac{\partial v^k}{\partial u^j} + \Gamma^k_{ij} v^i\right) w^j$. This expression for the covariant derivative is valid for any coordinate system with u^j as its variables. The expression is useful because it does not refer directly to the preferred coordinate system. All the required information is contained in the g_{ij} metric which will be in the expression denoted by Γ^k_{ij} . Note: $\delta_{\vec{w}}$ does not denote the Kronecker delta.

To develop the above expression, we start with a transition function between two charts $\vec{x}(\vec{u})$. The vector field $\vec{v}$ and the vector giving the direction of differentiation $\vec{w}$, are expressed in terms of u variables, with $\vec{v}\,'$ and $\vec{w}\,'$ expressed in x variables. The chart in x variables is euclidean in the $\vec{w}\,'$ direction.

Transforming back and forth in u coordinates gives:

$$\delta_{\vec{w}}\vec{v} = \left(\frac{\partial v'^m}{\partial x^p} w'^p\right)\frac{\partial u^k}{\partial x^m} = \left(\frac{\partial}{\partial x^p}\left(v^i \frac{\partial x^m}{\partial u^i}\right)\right)\left(\frac{\partial x^p}{\partial u^q} w^q\right)\frac{\partial u^k}{\partial x^m}$$

$$= \left(\frac{\partial v^i}{\partial u^j}\frac{\partial u^j}{\partial x^p}\frac{\partial x^m}{\partial u^i} + v^i \frac{\partial^2 x^m}{\partial u^i \partial u^j}\frac{\partial u^j}{\partial x^p}\right)\left(\frac{\partial x^p}{\partial u^q} w^q\right)\frac{\partial u^k}{\partial x^m}$$

$$= \left(\frac{\partial v^i}{\partial u^j}\frac{\partial x^m}{\partial u^i} + v^i \frac{\partial^2 x^m}{\partial u^i \partial u^j}\right) w^j \frac{\partial u^k}{\partial x^m}$$

This expresses the derivative in terms of u variables. But we don't want to directly reference the Cartesian chart. Here are highlights of the tricks used to remove the Cartesian chart reference.

We have $\delta_{\vec{w}}\vec{v} = \left(\frac{\partial v^i}{\partial u^j}\frac{\partial x^m}{\partial u^i} + v^i\frac{\partial^2 x^m}{\partial u^i \partial u^j}\right) w^j \frac{\partial u^k}{\partial x^m}$

We change its form by multiplying through by $\frac{\partial x^m}{\partial u^q}$, g^{kq}

and their inverses $\frac{\partial u^q}{\partial x^m}$, g_{kq}

:

$$\delta_{\vec{w}}\vec{v} = \left(\frac{\partial v^i}{\partial u^j}\frac{\partial x^m}{\partial u^i}\frac{\partial x^m}{\partial u^q} g^{kq} + v^i\frac{\partial^2 x^m}{\partial u^i \partial u^j}\frac{\partial x^m}{\partial u^q} g^{kq}\right) w^j \frac{\partial u^k}{\partial x^m}\frac{\partial u^q}{\partial x^m} g_{kq}$$

Then use the transformation rules of the metric from the euclidean metric to gather g_{iq} and g^{kq} terms.

$$\delta_{\vec{w}}\vec{v} = \left(\frac{\partial v^i}{\partial u^j} g_{iq} g^{kq} + v^i g^{kq}\frac{\partial^2 x^m}{\partial u^i \partial u^j}\frac{\partial x^m}{\partial u^q}\right) w^j g^{kq} g_{kq}$$

We finally express this in terms of g_{ij} by identifying one of the terms with the Christoffel symbol .

Use : $\frac{\partial^2 x^m}{\partial u^i \partial u^j}\frac{\partial x^m}{\partial u^q} = \frac{1}{2}\left(\frac{\partial g_{jq}}{\partial u^i} + \frac{\partial g_{qi}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^q}\right)$ (straight forward but busy)

Then recall $\Gamma^k_{ij} \equiv \frac{1}{2} g^{kq}\left(\frac{\partial g_{jq}}{\partial u^i} + \frac{\partial g_{qi}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^q}\right)$

to get us to the required derivative expression:

$$\delta_{\vec{w}}\vec{v} = \left(\frac{\partial v^k}{\partial u^j} + \Gamma^k_{ij} v^i\right) w^j$$

Components of the Embedded Derivative

We have defined the covariant derivative based on the rolling plane vision and it only depends on the metric.

In the case of a 2-d manifold embedded in euclidean 3-space, We can take the Cartesian derivative of the vector field. The vector representing the derivative can be expressed as a component perpendicular to the surface and a component in the tangent plane. The component in the tangent plane is the covariant derivative. More later when we study curvature.