8. 2-Form Polar Coordinate Transformation Example

We previously transformed basic 1-forms from Cartesian coordinates to polar coordinates and got:
$dx = \cos\theta\, dr - r\sin\theta\, d\theta$ and $dy = \sin\theta\, dr + r\cos\theta\, d\theta$

We will use wedge product algebra in the tangent plane to get the familiar looking $dx \wedge dy = r\, dr \wedge d\theta$

First, we expand the wedged parenthesis expressions the same way any multiplication is performed:
$dx \wedge dy = (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$

We get four terms. But we can reduce three of them by applying algebraic features of the wedge product like $d\theta \wedge d\theta = 0$, and $d\theta \wedge dr = -dr \wedge d\theta$

Of the four terms, two of them become zero, and one gets a sign change.

With these reductions, we get the expected result:

$$dx \wedge dy = r\sin^2\theta\, dr \wedge d\theta + r\cos^2\theta\, dr \wedge d\theta = r\, dr \wedge d\theta$$

Note that the wedge product behaves properly under transformation, $(dx \wedge dy)' = (dx)' \wedge (dy)'$. This follows from the tensorial nature of basic forms like $dx \wedge dy$ and the distributive law.

Levi-Civita Symbol and Determinants
We express k-forms in Einstein notation with k lower subscripts. For n-forms which are multiples of the determinant function, we can use a convenient tensor expression with n subscripts. You may recall that the determinant of a matrix can be expressed as the alternating sum of all the possible column products.

When working with expressions in Einstein notation there are technical expressions used as calculating tools. For example, we can guess the use of the Kronecker delta symbols δ^i_j and δ_{ij} which take the value of 0 or 1 depending on when the indices are equal. They can stand in for the identity matrix or unit basis vectors.

To express the determinant in Einstein notation, we use the Levi-Civita symbol, to assign $+$ or $-$ to a product according to permutations as shown below.

In 2-d the Levi-Civita symbol (or skew symmetric tensor) can be expressed as a matrix
$\epsilon_{ij} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$ and we can express $\det(\vec{a}, \vec{b})$ as

$$\begin{vmatrix} a^1 & b^1 \\ a^2 & b^2 \end{vmatrix} = \begin{pmatrix} a^1 & a^2 \end{pmatrix} \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \epsilon_{ij} a^i b^j$$

In 3-d the Levi-Civita symbol is $\epsilon_{ijk} = \begin{cases} -1 & \text{for odd permutations of ijk} \\ +1 & \text{for even permutations of ijk} \\ 0 & \text{otherwise} \end{cases}$

and $\det(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \epsilon_{ijk} a^i b^j c^k$

For n-d, we let σ be a permutation of the standard string of n indices

$$\epsilon_\alpha = \begin{cases} -1 & \text{for odd permutations of } \sigma \\ +1 & \text{for even permutations of } \sigma \\ 0 & \text{otherwise} \end{cases}$$

, where α is any sequence of the n indices.

and there is an analogous way (with tricky notation) to express the determinant for any square matrix.

Transforming an n-Form

We have seen that transforming a 1-form is identical to transforming a functional. Transforming a k-form on n-d space with 1<k<n can be complicated. We have seen that n-forms are especially simple because they are all multiples of the determinant.

We illustrate with a euclidean 3-d space with variables $\vec{x}$ and a parametrization $\vec{x}(\vec{u})$. The basic 3-form acting on 3 vectors is given by:

$$dx^1 \wedge dx^2 \wedge dx^3 (\vec{v}, \vec{w}, \vec{y}) = \det(\vec{v}, \vec{w}, \vec{y}) = \epsilon_{ijk} v^i w^j y^k$$

If we showed the details of the transformation of ϵ_{ijk} we would get that

$$\det(\vec{v}, \vec{w}, \vec{y}) = \left| \frac{\partial x^m}{\partial u^n} \right| \det(v'^i, w'^j, y'^k)$$

This is another way of describing how a determinant transforms under a change of basis.

Since we also have $\left| \frac{\partial x^m}{\partial u^n} \right|^2 = |g_{ij}|$, we can write:

$$dx^1 \wedge dx^2 \wedge dx^3 = \sqrt{|g_{ij}|}\, du^1 \wedge du^2 \wedge du^3$$

We will discuss $\pm\sqrt{|g_{ij}|}$ sign issues later.

For a general coordinate system change $\vec{y}(\vec{x})$, we get:

$$\sqrt{|g_{ij}|}dx^1 \wedge dx^2 \wedge dx^3 = \sqrt{|g_{ij}|}dy^1 \wedge dy^2 \wedge dy^3$$

where the g_{ij} 's are expressed in their home coordinate system variables.

In the context of polar coordinates where this operation is 2-d, and we get

$\left|\frac{\partial x^m}{\partial u^n}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$. And we confirm again that $dx \wedge dy \to r\,dr \wedge d\theta$.

Wedge Product Revisited

We have described differential forms in terms of various determinants. There is a combinatorial calculation that tells how to combine lower order differential forms into higher order forms expressed as wedge products. For differential forms α and β having orders p and q, we have

$$\alpha \wedge \beta \left(\vec{u}_{1,} \cdots \vec{u}_p, \vec{u}_{p+1}, \cdots \vec{u}_{p+q}\right) = \frac{1}{p!q!} \sum_{\sigma} (-1)^{\sigma} \alpha\left(\vec{u}_{\sigma_1}, \cdots \vec{u}_{\sigma_p}\right) \beta\left(\vec{u}_{\sigma_{p+1}}, \cdots \vec{u}_{\sigma_{p+q}}\right)$$

where the sum is over all permutations of length p+q.

This complicated expression results from the distributive law and the combining of terms as a result of wedge product anti-symmetry rules.

The wedge product of a p-form and a q-form results in a (p+q)-form or zero.

It is plausible that the wedge product behaves properly under transformations: $(\alpha \wedge \beta)' = \alpha' \wedge \beta'$, because the wedge product is built from simple linear operations on the underlying vector space.

Sometimes wedge product calculations are simple. Consider p and q forms $\alpha\left(\vec{u}_{1,} \cdots \vec{u}_p\right)$ and $\beta\left(\vec{v}_{1,} \cdots \vec{v}_q\right)$. If none of the $\vec{u}$'s are in the support of β then $\alpha \wedge \beta \;\left(\vec{u}_{1,} \cdots \vec{u}_p, \vec{v}_{1,} \cdots \vec{v}_q\right) = \alpha\left(\vec{u}_{1,} \cdots \vec{u}_p\right)\; \beta\left(\vec{v}_{1,} \cdots \vec{v}_q\right)$

Orientation

In the earth manifold example, there is a clear inside and outside to the surface. This is an orientable manifold. The euclidean plane is also orientable. Riemannian manifolds do not have sides but the concept is related to the chosen order of basis vectors in the tangent plane. Roughly speaking, an orientation of a manifold is a consistent choice of basis vectors in all the tangent planes. A manifold is orientable if there is an atlas where the Jacobian determinant of all the transition functions are positive. As a consequence, the representation of n-forms do not change sign on a transition between charts.

The Mobius strip is not orientable. Note that parallel transport of a vector once around the strip results in a mirror image flip of the vector. Typically, a non-orientable manifold will have a pair of charts where one (or any odd number) of the axes reverses direction.

For a non-orientable n-d manifold, there cannot be a continuous, never zero differential n-form. For this reason, we usually restrict discussions of differential forms to orientable manifolds. We simplify our work further by assuming that the Jacobian determinant of all transition functions are positive.

One-Forms and the Cotangent Plane

Consider a manifold defined by an embedding function. We described the basis vectors of the tangent plane as images of the standard basis vectors from the chart. In many presentations of differential geometry, the plane of covectors at a point provides the initial expression of the tangent space.

We demonstrate with polar coordinates on the xy-plane. On the xy-plane, there are two scalar valued functions $\theta(x,y)$ and $r(x,y)$ given by $\theta=\operatorname{atan}\left(\frac{y}{x}\right)$ and $r=\sqrt{x^2+y^2}$. The gradients of these scalar valued functions $\nabla\theta$ and ∇r are covector fields (covariant vector fields). The covectors are 1-forms and can be written as $d\theta$ and dr. This procedure is easy to apply in the context of a differentiable manifold. The basis covectors are determined by a set of scalar functions.

In the case of polar coordinates, consider two rotating fields:

$\frac{\partial(x,y)}{\partial\theta}$ the vector field (contravariant)

$d\theta$ the covector field (covariant)

You can distinguish the graphic representation of these two fields because the vector field increases in length away from the origin, and the covector field decreases in length away from the origin.

As a covector acting on a vector, we get the expected result:

$$d\theta\left(\frac{\partial(x,y)}{\partial\theta}\right)=1$$