7. Differential Forms

We continue with the theme of local linearity. You are familiar with expressing derivatives like $\frac{dy}{dx}$. The derivative expresses what the slope of a curve would be if it were a straight line. In this context, expressions like dy by itself do not have a clear meaning because of its infinitesimal intent. In many areas of mathematics and physics we write expressions like $\frac{dy}{dx}=f(x)$ as $dy=f(x)dx$, because it guides our infinitesimal thinking, even though it is not a traditional limit. This issue continues when we consider multiple variables in situations like polar coordinates $dx=\cos\theta\, dr - r\sin\theta\, d\theta$. Ultimately these differential expressions express local linearity.

So, $\Delta y = f(x)\Delta x \;\rightarrow\; dy=f(x)dx$ says that Δy is linearly dependent on Δx with linearity "constant" f(x) at small scales.

Integration and Changing Variables

We first encounter differential forms in the context of integration. Let's return to elementary calculus and recall a typical application of the substitution method for evaluating single variable integrals:

$$\int_0^R \sqrt{R^2-x^2}\,dx = \int_{\pi/2}^0 -R^2\sin^2\theta\, d\theta \quad \text{using} \quad x=R\cos\theta \qquad dx=-R\sin\theta\, d\theta$$

We will describe differential forms as tensor fields. Application of transition rules between charts is analogous to changing variables when integrating.

Consider attempting to use integration by substitution for an integrand with two variables like this polar coordinate example:

$$x=r\cos\theta \qquad y=r\sin\theta \quad \text{with} \quad dx=\cos\theta\, dr - r\sin\theta\, d\theta \;, \quad dy=\sin\theta\, dr + r\cos\theta\, d\theta$$

We might try:

$$\iint f(x,y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\big([\cos\theta\, dr - r\sin\theta\, d\theta][\sin\theta\, dr + r\cos\theta\, d\theta]\big) \quad ?$$

We could have a problem.

We will see how to handle multiple variable differential forms like dx dy. In this particular example we hope to end up with $dx\,dy = r\ dr\ d\theta$. Most people arrive at this result through geometrical insight involving circular sectors. We will show a general method of changing any number of integration variables without requiring specific geometrical insight.

Differential Forms and Surface Integration

We will examine differential forms with two variables. To this end we will consider surface integration. We will see why we view a differential form in two variables as a bilinear functional operating on pairs of vectors.

Consider integrand expressions like $f(x,y)\,dx\,dy$ over a region. We will denote the integrand as: $\omega = f(x,y)\,dx \wedge dy$. At first, this appears to just be a change of notation. We will see that the expression $dx \wedge dy$ (spoken as dx wedge dy) expresses a bilinear functional.

Consider the small rectangular shaped increments of area involved with integration, in terms of their spanning vectors. Integration involves a limiting sum like: $\sum f(x,y)\Delta x \Delta y \rightarrow \int f(x,y)\,dx\,dy$

Each rectangular increment is pinned at one corner to the origin of the local coordinate system. With this pinned leaf picture in mind, we rewrite the above sum as: $\sum f(x,y)\det(\Delta x\,\hat{x}\ ,\ \Delta y\,\hat{y})$

We can view the expression $f(x,y)\det(\Delta x\,\hat{x}\ ,\ \Delta y\,\hat{y}) = \omega(\Delta x\,\hat{x}\ ,\ \Delta y\,\hat{y})$ as a bilinear functional operating on pairs of small but finite vectors in the local coordinate system.

We call $\omega = f(x,y)\,dx \wedge dy$ a differential form. It is a special bilinear functional (doubly covariant tensor field) that we define in terms of the area spanned by the vectors. In our example, the region to be integrated was divided into increments that are spanned by vectors aligned with the local coordinate system. The wedge part of the notation expresses the variables involved. The f(x,y) part assigns the local weight to the integration increment.

The same differential form can be applied to any dissection of the region into approximate parallelograms, each spanned by vectors $\Delta\vec{u}$ and $\Delta\vec{v}$ and we can write:

$$\int f(x,y)\,dx \wedge dy \approx \sum f(x,y)\,(dx \wedge dy)(\Delta\vec{u}\ ,\ \Delta\vec{v}) = \sum f(x,y)\det(\Delta\vec{u}\ ,\ \Delta\vec{v})$$

where $\Delta\vec{u}$ and $\Delta\vec{v}$ are expressed in terms of the basis $\hat{x}$ and $\hat{y}$.

So the expression $dx \wedge dy$ does not directly express an infinitesimal concept, but it guides the linear limiting process on arbitrarily small increments

As a consequence of its relationship to the determinant, the order of the variables in the expression $dx \wedge dy$ expresses a 2-d orientation. In particular, $dy \wedge dx = -dx \wedge dy$. This skew symmetry results from swapping columns in the determinant. Skew symmetry of differential forms corresponds to the alternating nature of determinants.

In general, we express an integrand in terms of a field of bilinear functionals (or multi-linear functionals). We can use tensor methods to go between charts. The skew symmetric property of the determinant operation gives a way to express orientation.

In our context we define a differential form to be a skew symmetric multilinear functional field. It must be a multiply covariant tensor field so that its transformed version yields the same result when applied to a list of transformed vectors.

We identify the order of a differential form by noting the number of vectors that it acts on. For example:

$\omega(\vec{u},\vec{v})=dx\wedge dy(\vec{u},\vec{v})$ is a 2-form
$\omega(\vec{v})=f(x,y,z)\ dy(\vec{v})$ is a 1-form
$\omega()=g(x,y,z)\ ()=g(x,y,z)$ is a 0-form

The dimension of the underlying vector space limits the set of differential forms. There are no 4-forms on the space of 3-d vectors.

Also, the skew symmetry condition also restricts the set of multi-linear functionals that can be differential forms. In the case of n-forms on the space of n-d vectors, the only n-forms are multiples of the determinant. For an n-d vector space, the k-forms with 1<k<n are a little more complicated.

Wedge Notation and Differential Forms on 3-d space

Consider 3-d Cartesian space. We take the expression $\omega=dx^1\wedge dx^2\wedge dx^3$ to be the determinant operation on three vectors. Still within 3-d Cartesian space, the expression $\omega(\vec{u},\vec{v})=dx\wedge dy(\vec{u},\vec{v})$ signifies the area spanned by $\vec{u}$ and $\vec{v}$ after projection onto the xy-plane. Note that $dx\wedge dx(\vec{u},\vec{v})=0$ because the xx-plane is degenerate. Similarly $dx(\vec{u})$ is the length of $\vec{u}$ after projection onto the x-axis. The variables in a wedge product designate the subspace supporting it. We can still use the 3x3 determinant to calculate k-forms with k <3 by inserting dummy standard basis vectors into the determinant expression. For example, if $\omega=dx^1\wedge dx^2$ then $\omega(\vec{u}\ ,\ \vec{v})=\det(\vec{u}\ ,\ \vec{v}\ ,\ \hat{x}_3)$.
Similarly $(dx^2\wedge dx^3)(\vec{u}\ ,\ \vec{v})=\det(\hat{x}_1\ ,\ \vec{u}\ ,\ \vec{v})$. Note that $(dx^2\wedge dx^3)(\vec{u}\ ,\ \vec{v})=0$ if $\vec{u}$ or $\vec{v}$ is orthogonal to the $(\hat{x}_2,\hat{x}_3)$ plane. We continue the scheme with a 1-form acting on a single vector $(dx^2)(\vec{u})=\det(\hat{x}_1\ ,\ \vec{u}\ ,\ \hat{x}_3)=u^2$, so that dx^2 is the functional that extracts the second coordinate.

We looked at wedge products of basic forms like $dx\wedge dy$ and $(dx\wedge dy)\wedge dz$. We extend the algebra of the wedge product via linearity, the distributive law and skew symmetry (like vector cross product).

Coordinate Transformations and 1-Forms

We have expressed the concept of differential forms in terms of tensors and can extend the concept to Riemannian manifolds. A 1-form is another name for a covector field. In differential form notation, the coefficients of dx, dy, and dz are the components of each covector.

Suppose we have a coordinate transformation like $\vec{x}(\vec{u})$ and we wish to transform a 1-form like dx^i .This is another notation for the particular functional (covector) that extracts the i^{th} component from a vector. When changing from the $\vec{x}$ coordinate system to the $\vec{u}$ coordinate system, we apply the rule for a covariant tensor. When this is expressed in differential form notation, we get:

$$dx^i \rightarrow \frac{\partial x^i}{\partial u^k} du^k$$

This has the same appearance as the familiar integral calculus infinitesimal expression after changing variables.

If we apply this to polar coordinates, we get differential form expressions:

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta \, dr - r\sin\theta \, d\theta$$ (remember to sum !)

and $$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta \, dr + r\cos\theta \, d\theta$$

Here equality means, giving the same result when applied to a transformed vector. We can see how these covector expressions show local linearity with

$$dx\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = \cos\theta \, dr\begin{pmatrix}\Delta r \\ \Delta\theta\end{pmatrix} - r\sin\theta \, d\theta\begin{pmatrix}\Delta r \\ \Delta\theta\end{pmatrix}$$

for any pair of corresponding vectors $\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix}$ and $\begin{pmatrix}\Delta r \\ \Delta\theta\end{pmatrix}$ in the respective coordinate systems.

When we apply the component picking features of dx, dr, and $d\theta$, we get $\Delta x = \cos\theta \, \Delta r - r\sin\theta \, \Delta\theta$ .We will use this later

So, we have expressed infinitesimal concepts in terms of local linearity. These ideas will be clearer when we revisit $dx\,dy \rightarrow r\,dr\,d\theta$