6. Mobius Strip Example

This is a way of representing the manifold obtained from a long paper rectangle by attaching the ends after applying a half twist. In this presentation, three 10 cm strips are used in order to isolate the transition mappings. The manifold can be constructed from three charts:

C1: $-5 \le x^1 \le 5 \quad -1 \le x^2 \le 1$
C2: $-5 \le y^1 \le 5 \quad -1 \le y^2 \le 1$
C3: $-5 \le z^1 \le 5 \quad -1 \le z^2 \le 1$

The transition mappings on the single unit overlaps are described by:

T12: $y^1 = x^1 - 9$ with $y^2 = x^2$ for $4 \le x^1 \le 5$
T23: $z^1 = y^1 - 9$ with $z^2 = -y^2$ for $4 \le y^1 \le 5$
T31: $x^1 = z^1 - 9$ with $x^2 = y^2$ for $4 \le z^1 \le 5$

The half twist is represented by T23. There is no reference to a potentially complicated embedding. The pieces constructing the manifold are locally flat, so we choose Its Riemannian metric g_{ij} to be the 2-d euclidean metric on all three charts.

Later, we will study parallel transport. As an introduction, consider a unit vector oriented upward across the strip at the center of C1. We then move this vector to one end of the chart keeping its same upward orientation (parallel transport). Then continue moving the vector in the obvious way from chart to chart through the overlaps. After traversing the strip, we arrive at the starting point and find that the vector has become inverted. On the paper model, it will be on the opposite side, but intrinsic geometry does not distinguish between sides in this case.

Hamilton's Principle

We can express the arc length along a path between two points in a chart as:

$$\int_{\vec{a}}^{\vec{b}} \sqrt{\frac{dx^i}{dt}\frac{dx^j}{dt} g_{ij}}\, dt$$

We wish to find the path with the shortest arc length between $\vec{a}$ and $\vec{b}$ (geodesic).

To this end, we digress into a special topic involving integrals of the type: $\int_{t_1}^{t_2} L(\vec{x}, \vec{y})\, dt$ with a further restriction that $\vec{y}(t) = \frac{d}{dt}\vec{x}(t) = \dot{\vec{x}}(t)$. These integrals are also called action integrals.

In slightly confusing notation we will write some derivatives of L as:

$\frac{\partial L}{\partial \dot{x}^i}$ to mean $\frac{\partial L}{\partial y^i}$ at $\vec{y}(t)=\dot{\vec{x}}(t)$

Hamilton's principle is a method for minimizing path integrals.

Let an action integral along a path $\vec{x}(t)$ with end points $\vec{x}(t_1)=\vec{a}$ and $\vec{x}(t_2)=\vec{b}$ be given by: $A(\vec{x},\dot{\vec{x}})=\int_{t_1}^{t_2} L(\vec{x},\dot{\vec{x}})dt$.

We have a theorem that, a path $\vec{x}(t)$ that minimizes $A(\vec{x},\dot{\vec{x}})$ satisfies the Euler-Lagrange differential equation:

$\frac{\partial L}{\partial x^i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}=0$ and vice versa.

We show the main tricks of the proof in order to show its flavor. It is not important that you follow the details.

Consider an optimal path $\vec{x}(t)$ and a perturbation of the path expressed as $\vec{y}(\alpha,t)=\vec{x}(t)+\alpha\,\vec{\eta}(t)$. Here $\vec{\eta}$ is any fixed but arbitrary path with $\vec{\eta}(t_1)=\vec{0}=\vec{\eta}(t_2)$ and α is a parameter going to zero as we reduce the perturbation.

An extremal path satisfies the zero derivative relation:

$$0=\frac{d}{d\alpha}A(\vec{y},\dot{\vec{y}})\Big|_{\alpha=0} = \lim_{\alpha\to 0}\frac{A(\vec{x}+\alpha\vec{\eta},\dot{\vec{x}}+\alpha\dot{\vec{\eta}})-A(\vec{x},\dot{\vec{x}})}{\alpha} \quad \text{for any } \vec{\eta}$$

Calculating the α derivative of the action integral gives (skipping a few steps):

$$0=\frac{d}{d\alpha}A(\vec{y},\dot{\vec{y}})=\int_{t_1}^{t_2}\frac{d}{d\alpha}L(\vec{x}+\alpha\vec{\eta},\dot{\vec{x}}+\alpha\dot{\vec{\eta}})dt$$

$$=\int_{t_1}^{t_2}\frac{\partial L}{\partial x^i}\eta^i+\frac{\partial L}{\partial \dot{x}^i}\dot{\eta}^i dt=\int_{t_1}^{t_2}\frac{\partial L}{\partial x^i}\eta^i dt+\left[\frac{\partial L}{\partial \dot{x}^i}\eta^i\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\eta^i dt$$

$$=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x^i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\eta^i dt \quad \text{for any } \vec{\eta}$$

so $0=\frac{\partial L}{\partial x^i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}$ for all i.

And the implications reverse so that, if a path $\vec{x}(t)$ satisfies the Euler-Lagrange equation then it is an extremal path for the action integral $A(\vec{x},\dot{\vec{x}})$.

Geodesics

Our basic understanding of geodesics comes from finding the path having the shortest distance between two points on the surface of the earth. We know these as portions of great circles. The image of a great circle in the Atlantic chart is serpentine. On a polar chart it is an arc. We will develop the differential equations that a geodesic must satisfy in a general chart where g_{ij} is given. This tool can be used to find geodesics on more complicated surfaces such as an egg carton.

We can express the arc length along a path between two points in a chart as:

$$\int_{\vec{a}}^{\vec{b}} \sqrt{\frac{d\,x^i}{dt}\frac{d\,x^j}{dt} g_{ij}}\, dt$$

We wish to find the path with the shortest arc length between $\vec{a}$ and $\vec{b}$ (geodesic).

We are given the metric g_{ij} and we wish to minimize the arc length given by

$$\int_{\vec{a}}^{\vec{b}} \sqrt{\dot{x}^i g_{ij} \dot{x}^j}\ dt = \int_{\vec{a}}^{\vec{b}} L(\vec{x}, \dot{\vec{x}})\ dt \quad \text{using dot notation for} \quad \frac{d}{dt}$$

so, $\vec{x}$ and $\dot{\vec{x}}$ satisfy $0 = \dfrac{\partial L}{\partial x^q} - \dfrac{d}{dt}\dfrac{\partial}{\partial \dot{x}^q} L$ for all q.

We show proof highlights to show the general idea, skipping details. The variables and dotted variables in the domain of $L(\vec{x}, \dot{\vec{x}})$ are all independent until we take the t derivative. So derivatives between these variables are usually zero (and one, when they are the same). We express the derivative of the three-fold product in the integrand using Einstein notation and reduce the three terms, using independence.

So, $$\frac{\partial}{\partial \dot{x}^q}(\dot{x}^i g_{ij} \dot{x}^j) = \frac{\partial \dot{x}^i}{\partial \dot{x}^q} g_{ij} \dot{x}^j + \dot{x}^i\ \frac{\partial g_{ij}}{\partial \dot{x}^q}\ \dot{x}^j + \dot{x}^i\ g_{ij}\ \frac{\partial \dot{x}^j}{\partial \dot{x}^q}$$

$$= g_{qj} \dot{x}^j + \dot{x}^i\ 0\ \dot{x}^j + \dot{x}^i\ g_{iq} = 2\, g_{iq} \dot{x}^i$$

We then can express the Euler-Lagrange equation in Einstein notation as

$$0_q = \frac{1}{2L}\ \frac{\partial g_{ij}}{\partial x^q}\ \dot{x}^i \dot{x}^j - \frac{d}{dt}\frac{1}{L}\ g_{iq} \dot{x}^i$$

This becomes simpler when we assume parametrization is by arc length. A little work (which is omitted here) results in L=1 along the path.

So, $0_q = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^q}\dot{x}^i\dot{x}^j - \frac{d}{dt}g_{iq}\dot{x}^i$

Apply the chain rule to get $\frac{d}{dt}g_{iq}\dot{x}^i = \frac{\partial g_{iq}}{\partial x^j}\dot{x}^i\dot{x}^j + g_{iq}\ddot{x}^i$

Use symmetry of g_{ij} to get $\frac{\partial g_{iq}}{\partial x^j}\dot{x}^i\dot{x}^j = \frac{\partial g_{qj}}{\partial x^i}\dot{x}^i\dot{x}^j$

and get $0_q = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^q}\dot{x}^i\dot{x}^j - \frac{\partial g_{iq}}{\partial x^j}\dot{x}^i\dot{x}^j - g_{iq}\ddot{x}^i = \frac{1}{2}\left(-\frac{\partial g_{ji}}{\partial x^q} + 2\frac{\partial g_{qj}}{\partial x^i}\right)\dot{x}^i\dot{x}^j + g_{iq}\ddot{x}^i$

Use a summation symmetry argument to get

$$\left(\frac{\partial g_{iq}}{\partial x^j} + \frac{\partial g_{jq}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^q}\right)\dot{x}^i\dot{x}^j = \left(2\frac{\partial g_{iq}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^q}\right)\dot{x}^i\dot{x}^j$$

and get $0_q = g_{iq}\ddot{x}^i + \frac{1}{2}\left(\frac{\partial g_{iq}}{\partial x^j} + \frac{\partial g_{jq}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^q}\right)\dot{x}^i\dot{x}^j$

so that $0^k = g^{kq}g_{iq}\ddot{x}^i + \frac{1}{2}g^{kq}\left(\frac{\partial g_{iq}}{\partial x^j} + \frac{\partial g_{jq}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^q}\right)\dot{x}^i\dot{x}^j$

Here, g^{kq} is defined as the inverse of g_{iq} so that $g^{kq}g_{iq} = \delta_i^k$

This reduces to the differential equations for geodesic paths that are parametrized by arc length:

$0^k = \ddot{x}^k + \Gamma_{ij}^k\dot{x}^i\dot{x}^j$

where $\Gamma_{ij}^k \equiv \frac{1}{2}g^{kq}\left(\frac{\partial g_{jq}}{\partial x^i} + \frac{\partial g_{qi}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^q}\right)$ also denoted as $\left\{\begin{matrix} k \\ i\ j \end{matrix}\right\}$, is the Christoffel symbol, which usually is not a tensor because of the chart dependent g_{ij} derivatives.

We will see the Christoffel symbol again when we present covariant differentiation.

Note that the differential equation for geodesics is consistent with straight lines in euclidean space where all the derivatives of the euclidean metric are zero.