5. Tensor Fields

We saw that, a field of vectors or functionals (co vectors) defined in terms of one parameterization is systematically defined in any other parametrization covering the same point.

In the realm of parametrized surfaces, a typical tensor field is a vector field or co-vector field defined in the tangent planes of a surface. It is usually denoted by its coordinates at each point in a parameter space using Einstein summation notation. The same vector field or co-vector field can be expressed in other parameterizations by using the implied transformation rules. Of course we need to have the transition functions between parameterizations in order to do the required linear transformations. We are frequently lax in distinguishing between tensors in a single tangent plane and tensor fields.

We must distinguish between vectors and functionals so that we know how to transform them. The upper and lower indices of Einstein notation signify this distinction. Historically vectors are called contravariant tensors and co-vectors are called covariant tensors (covariant and contravariant can have reverse meaning depending on context!). The term covariant in this context is not related to the covariance idea in probability theory.

Tensors of higher order (or degree or rank) can be other linear objects with multiple indices such as matrices or higher dimensional arrays. They can be multiply covariant, contravariant or mixed, depending on their intended application. When mixed tensors change variables, they use analogous transformation rules, being multiplied several times by the Jacobian matrix of the transition function as necessary, according to their covariant or contravariant application.

For example g_{ij} is doubly covariant, giving the same result when applied to transformed vectors. Consider vectors a^i and b^j in the parameter space of $\vec{y}$ variables and their corresponding transformed representation in the parameter space of $\vec{x}$ variables $a^m \frac{\partial y^i}{\partial x^m}$ and $b^n \frac{\partial y^j}{\partial x^n}$

Then g_{ij} works as a doubly covariant tensor because the expression

$\left(a^m \frac{\partial y^i}{\partial x^m}\right) g_{ij} \left(b^n \frac{\partial y^j}{\partial x^n}\right)$ in the space of $\vec{x}$ variables

becomes $a^m \left(g_{ij} \frac{\partial y^i}{\partial x^m} \frac{\partial y^j}{\partial x^n}\right) b^n$ in the space of $\vec{y}$ variables.

A zero order tensor field is a scalar valued function. Its transformation rule is just the simple substitution operation between parameterizations and is rarely mentioned.

A tensor field can be defined by an algorithm to be applied in any parameter space and then verified to obey transformation rules. Typically the algorithm depends on values at the point or tangent plane. Or, a tensor may be defined in one parametrization (expressed in Einstein notation) and then propagated to all other overlapping parameterizations via the implied transformation rules. There will be more exhibits of the role of the transformation rules Later.

The main feature of a field of linear objects that makes it a tensor field, is its implied transition rules for its expression in other parameterizations without reference to the embedding of the surface. Often, derivatives of tensor fields aren't tensor fields. However, we have seen that the gradient of a scalar field is a covariant tensor.

Differentiable Manifolds
A differentiable manifold is a surface (not necessarily 2-d) that is covered by an overlapping collection of differentiable parameterizations. In this context, parameterizations are called charts, and the collection of charts is called an atlas.

There are further requirements on the charts. They must be one-to-one. Their domains must be Cartesian spaces with the same dimension. The transition mappings between charts on their overlap must be one-to-one and differentiable. If the surface has edges, they will be represented in the charts and they must match where the charts overlap.

In the geography example, we have already specified one chart, that has a discontinuity at the international dateline. We can complete the atlas, with a similar chart having its discontinuity along the Greenwich meridian, and two polar projections.

The g_{ij} tensor has special significance because of its role in integration and curvature.

Earth Manifold Example
We describe an atlas for the earth.

The Atlantic chart was described previously and is centered on the equator and Greenwich meridian :

$$x = R\cos\alpha\cos\beta \quad , \quad y = R\sin\alpha\cos\beta \quad , \quad z = R\sin\beta \quad \text{where} \quad (\text{lon}, \text{lat}) = (\alpha, \beta)$$
$$(-180^\circ < \alpha < 180^\circ) \qquad (-90^\circ < \beta < 90^\circ)$$

The Pacific chart is similarly centered near the Polynesian islands with a similar definition:

$$x = R\cos\alpha\cos\beta \quad , \quad y = R\sin\alpha\cos\beta \quad , \quad z = R\sin\beta \quad \text{where} \quad (\text{lon}, \text{lat}) = (\alpha, \beta)$$
$$(0^\circ < \alpha < 360^\circ) \qquad (-90^\circ < \beta < 90^\circ)$$

The North chart is a simple projection with x and y variables being both chart variables and embedding space variables.

$$x = x \ \ , \ \ y = y \ \ , \ \ z = +\sqrt{R^2 - x^2 - y^2} \ \ \text{where} \ \ x^2 + y^2 < R^2$$

The South chart is similarly defined with

$$x = x \ \ , \ \ y = y \ \ , \ \ z = -\sqrt{R^2 - x^2 - y^2} \ \ \text{where} \ \ x^2 + y^2 < R^2$$

An Easterly Vector and Co-Vector Field

We exhibit tensor terminology with a unit vector field pointing east on the surface of the earth (undefined at the poles).

We bypass the algebra and sketch the vector fields in the Atlantic and north charts. In the Atlantic chart, the vector field appears to point to the right with larger magnitudes toward the poles. In the north chart, the vector field appears to circulate counter clockwise around the north pole with a uniform magnitude. We note that there is a transition function from the Atlantic chart to the north chart given by $x = R \cos\alpha \cos\beta$ and $y = R \sin\alpha \cos\beta$

The vector field looks quite different in the two charts, but we know they are the same because they both come from the same source in the tangent planes of the earth. We say that the two sketches represent the same tensor field because their representations in the two charts follow the the contravariant transformation rule. Ultimately, there is one underlying vector field with multiple representations.

If the east pointing flow is deemed to be a co-vector field, Its representation in the Atlantic chart will be changed so that its magnitude decreases toward the poles. This is a manifestation of the inverse transformation rule for functionals so that its action on vectors is conserved. The sketch of the co-vector field in the north chart is the same as the sketch of the vector field.

The use of tensors and Einstein notation give a systematic method for changing variables between charts.

Riemannian Geometry

Consider the case of a collection of charts without the embedding parametrization, but with transition mappings where they overlap. We can still understand much of the geometry of the manifold because the charts and transition mappings give a patchwork construction plan like a dressmakers pattern. If a manifold is defined by an initial covering atlas, all other charts are constrained by the transition mapping requirement.

Our concept of tensor fields still applies. Furthermore, if we are given the g_{ij} metric tensor, we have sufficient information to perform many differential geometry tasks, such as surface integration and identifying geodesics. This is called Riemannian geometry because it uses the Riemannian metric g_{ij} . This is a special type of intrinsic geometry, meaning that it depends on properties of the surface, without reference to the embedding. We call g_{ij} a metric because of its use in calculating arc length.

Intrinsic geometry is an attractive tool for general relativity because space curvature geometry can be expressed in terms of measurements within our local physical space. Each locality in space is identified with a chart. With general relativity applications, the 4-d g_{ij} is not positive definite.

Ordinary euclidean space with a Cartesian chart is a simple manifold. We usually take g_{ij} to be the identity matrix and call it the euclidean metric.

Riemannian Geometry of Earth

We have the Riemannian geometry of the earth via its four charts with reference to its 3d embedding. However, we need the expressions for the Riemannian metric and the transition functions. We examine the Atlantic and north charts.

The transition function from the Atlantic chart to the north chart is given by
$x = R\cos\alpha\cos\beta$, $y = R\sin\alpha\cos\beta$ with derivatives

$$\frac{\partial x}{\partial\alpha} = -R\sin\alpha\cos\beta \qquad \frac{\partial x}{\partial\beta} = -R\cos\alpha\sin\beta$$

$$\frac{\partial y}{\partial\alpha} = R\cos\alpha\cos\beta \qquad \frac{\partial y}{\partial\beta} = -R\sin\alpha\sin\beta$$

We previously noted that the Riemannian metric for the Atlantic chart works out to $g_{A\,ij}(\alpha,\beta) = \begin{bmatrix} R^2\cos^2\beta & 0 \\ 0 & R^2 \end{bmatrix}$.

The Riemannian metric for the north chart works out to $g_{N\,ij} = \begin{bmatrix} \frac{R^2-y^2}{z^2} & \frac{xy}{z^2} \\ \frac{xy}{z^2} & \frac{R^2-x^2}{z^2} \end{bmatrix}$

where z is given by $R^2 = x^2 + y^2 + z^2$.

With some algebraic effort we could show that the Riemannian metrics are related by the tensor transformation rule:

$g_{A\,ij} = \frac{\partial x^p}{\partial\alpha^i}\frac{\partial x^q}{\partial\alpha^j} g_{N\,pq}$ with the notation $(x,y) = (x^1, x^2)$ and $(\alpha,\beta) = (\alpha^1, \alpha^2)$.