1. Naive Differential Geometry

Differential geometry is based on the study of curved surfaces. We consider questions like finding the shortest path within a surface (geodesic), and how to work with derivatives and other linear concepts (tensors) on a surface. We will show how to work with curved surface geometry as though we were working with a flat surface.

The historical evolution of differential geometry concepts is complex. We mostly follow the view of the early 20^{th} century. The modern development is more elegant but requires mathematical sophistication. We constrain most of our discussion to a level of calculus and linear geometry encountered in the first two years of a university or college education. This is a prequel to Riemannian geometry. We omit most of the proofs and are lax in our notation and definitions in order to concentrate on the direction of thought.

We will explore tensors, geodesics, differential forms, and curvature.

Differential geometry has its roots in the study of smooth surfaces. The smooth property leads us to consider surface derivatives and other calculus issues.

A simple surface can be expressed as a function of two variables such as the parabolic bowl $z=f(x,y)=x^2+y^2$

Parameterization

A more general surface can be expressed through a parameterization like the familiar geographic coordinates for the earth surface
$x=R\cos\alpha\cos\beta$, $y=R\sin\alpha\cos\beta$, $z=R\sin\beta$ where $(\text{lon},\text{lat})=(\alpha,\beta)$
The equatorial plane is represented as $(x,y,0)$. The x axis pierces the Atlantic ocean near Africa and the y axis pierces the Indian ocean near Java.

Here the earth has radius R. The parametrization works except at the poles and the seam near the international date line $(-180^\circ<\alpha<180^\circ)$. Additional overlapping parameterizations (charts) are required to describe the entire earth. This is similar to the multiple projections found in an atlas.

In this case, the parameter space (of α,β) is a 2-d Cartesian space but the embedding space (of x,y,z) is a 3-d Cartesian space.

We know that a geodesic on the earth is a great circle. How do we find a geodesic in the parameter space? We will find its differential equation.

You may recall changing variables when you learned integration. Changing variables for a surface is equivalent to re-parametrization. You may recall changing variables to polar coordinates $x=r\cos\theta$, $y=r\sin\theta$.
Later we will address the issue of $dx\,dy \rightarrow r\,dr\,d\theta$

Linear Algebra
The study of linear algebra concentrates on simple functions that are linear. In one variable, a linear function is a line through the origin $f(x)=mx$. A linear function of two variables is a plane through the origin $f(x,y)=ax+by$. Here, we express a list of variables as a vector and allow ourselves to work with vector valued functions $\vec{y}=\vec{f}(\vec{x})$. Any vector valued linear function applied to a vector can be expressed as a matrix applied to the vector.

In the case of $\vec{c}=\vec{f}(\vec{b})$ we can find a matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and write:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \vec{c}=A\vec{b} \qquad \left(\text{note} \quad \begin{matrix} c_1=a_{11}b_1+a_{12}b_2 \\ \text{and} \\ c_2=a_{21}b_1+a_{22}b_2 \end{matrix}\right)$$

We interchangeably use the terms: transformation matrix, and linear transformation.

In the case of a scalar valued linear function $c=f(\vec{b})$ we can find a row vector $(a_1\ ,\ a_2)$ and write

$$c = f(\vec{b}) = (a_1\ ,\ a_2)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1+a_2 b_2$$

A scalar valued linear function (especially simple) is called a functional or equivalently a co-vector. It is represented as a row vector. The space of row vectors is called the dual space of the vector space. We will revisit the distinction between vectors and functionals (co-vectors) in different contexts. Eventually, this distinction will become clear.

A bilinear functional is a scalar valued linear function of two vectors $s=f(\vec{a},\vec{c})$

Matrices can also represent a bilinear functional:

$$(a_1\ ,\ a_2)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=s \qquad \vec{a}^t\, B\, \vec{c}=s$$

Note the use of superscript t denoting transpose, to represent a row vector.

When we discuss integration, we will calculate the area of a parallelogram spanned by two vectors. We can form a matrix where the vectors are the columns. The determinant of this matrix is the required area.

$$\text{Area}=\det(\vec{a},\vec{b})=\left|\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right|=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1 b_2-a_2 b_1$$

The area is oriented (signed) according to the permutation of the vectors. This determinant concept generalizes to higher dimensions such as the parallelepiped volume spanned by three vectors.

Area Spanned by Two Vectors in a Space with Larger Dimension

When two vectors span a parallelogram in 3-d space, we cannot determine the area using the determinant. We could change the coordinate system so that the vectors lie in a coordinate axis plane. We could use the cross product, but this technique is not general enough for larger dimension spaces.

We note that the area squared, in 2-d space is given by

$$\text{area}^2 = \det^2(\vec{a}, \vec{b}) = |C^t||C| = |C^t C| \quad \text{where} \quad C = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The matrix $C^t C = \begin{bmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{bmatrix}$ is the square matrix of column dot products.

The determinant $|C^t C|$ takes the same value regardless of embedding in higher dimension spaces. This expression is sometimes called Lagrange's identity. We will encounter this construction again on the way to understanding Riemannian geometry.

Changing Basis

So far, we have represented vectors as lists of scalar coordinates. We have implicitly assumed a standard basis like the familiar $[\hat{x}, \hat{y}, \cdots]$

We will be expressing vectors in an alternative basis like $[\vec{u}, \vec{v}]$. The coordinate expressions for a vector $\vec{a}$ in the two bases an be written as:

$$\vec{a} = a_1 \hat{x} + a_2 \hat{y} = a_1^{'} \vec{u} + a_2^{'} \vec{v}$$

To find the relationship between the standard and transformed coordinates, we create the transformation matrix mapping the standard basis into the alternative basis. Note that $\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

This leads us to consider the matrix whose columns are the alternative basis elements:

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = T \quad \text{resulting in} \quad \vec{u} = T \hat{x} \quad \text{and} \quad \vec{v} = T \hat{y}$$

This extends to $\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{pmatrix} a_1^{'} \\ a_2^{'} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ expressed as $\vec{a} = T \vec{a}^{'}$

Furthermore we have $\vec{a}' = T^{-1}\ \vec{a}$. This tells us how to transform coordinates of a vector expressed in an alternate basis.

Transforming a Functional

When we express a functional in an alternative basis, we would like its effect on vectors to be conserved through the transformation, so that $\left(\vec{a}^t\right)'\ \vec{x}' = \vec{a}^t\ \vec{x}$, where $\vec{x}' = T^{-1}\ \vec{x}$.

The rule for transforming functionals must be $\left(\vec{a}^t\right)' = \vec{a}^t T$
because $\vec{a}^{t'}\ \vec{x}' = \left(\vec{a}^t T\right)\left(T^{-1}\vec{x}\right) = \vec{a}^t\ \vec{x}$.

We will usually express these dual transformation rules in the form

$$\vec{x} = T\ \vec{x}' \quad \text{and} \quad \left(\vec{a}^t\right)' = \vec{a}^t T$$

Again we show the invariance of the result, when applying a functional to a vector when changing bases. Here, we use the common form of the transformation rules.

$$\left(\vec{a}^t\right)'\ \vec{x}' = \left(\vec{a}^t T\right)\vec{x}' = \vec{a}^t\left(T\,\vec{x}'\right) = \vec{a}^t\ \vec{x}$$

The dual transformation rules are part of the reason why we distinguish between a vector space and its dual space of functionals.

Transforming Other Linear Quantities

Analogous transformation rules for other linear objects are designed to preserve the result of the operations that they imply. As a further example consider a transformation matrix or a bilinear functional A.

Its transformation rule should be $A' = T^{-1} A\, T$, where the columns of T are the new basis vectors.

$$\text{because} \quad A'\ \vec{b}' = \left(T^{-1} A\, T\right)\left(T^{-1}\vec{b}\right) = \left(A\,\vec{b}\right)'$$

$$\text{and} \quad \left(\vec{b}^t\right)'\ A'\,\vec{c}' = \vec{b}^t T\left(T^{-1} A\, T\right)\left(T^{-1}\vec{c}\right) = \vec{b}^t A\,\vec{c}$$