Understanding the Riemann Curvature Tensor

The Riemann Curvature Tensor is key to the definition of Riemannian curvature. I have been overwhelmed by its complexity and wondered how anyone would discover it. Historically it emerged from considerations of the derivatives of the Riemannian metric. I want to share a possible false history explanation that improved my vision of this tensor. To understand my remarks, you should be acquainted with covariant differentiation, differential forms, exterior differentiation, the general stokes theorem, and Riemannian curvature.

Suppose you wanted to understand curvature by looking at the change in a vector field around a small loop. To this end, you might want to integrate the derivative of the vector field around the loop. This would be $\vec{0}$ in a Euclidean space. For a closed path $s(t)$, this would look like $\oint \nabla_{\dot{s}} \vec{V} \ dt$ .Here, $\nabla_{\dot{s}} \vec{V}$ is calculated by applying the Jacobian matrix of $\vec{V}$ to the derivative of the path s.

When doing an integral, we think of differential forms. In this case we have a vector valued form like $\vec{\omega}(\vec{u}) = \nabla_{\vec{u}} \vec{V}$. We are in a curved space and want our derivative expression to be a tensor. So we should use the covariant derivative and define our form as $\vec{\omega}(\vec{u}) \equiv \delta_{\vec{u}} \vec{V}$.

We will be wanting to compare our line integral with the area enclosed by the loop. To this end, we would like to invoke stokes theorem and convert our loop integral to an area integral of the exterior derivative like, $\oint \vec{\omega} = \iint\limits_{\text{loop area}} d\vec{\omega}$. This is similar to converting a loop integral of a vector field to an area integral of the curl of the vector field.

We have a convenient formula for calculating the exterior derivative of a scalar valued 1-form, $d\omega(\vec{U},\vec{W}) = \nabla_{\vec{U}}\ \omega(\vec{W}) - \nabla_{\vec{W}}\ \omega(\vec{U}) - \omega([\vec{U},\vec{W}])$, where $[\vec{U},\vec{W}]$ is the Lie bracket defined as $[\vec{U},\vec{W}] \equiv \nabla_{\vec{U}}\vec{W} - \nabla_{\vec{W}}\vec{U}$. We can use a similar expression for the exterior derivative of our vector valued form:

$$d\vec{\omega}(\vec{U},\vec{W}) = \delta_{\vec{U}}\ \vec{\omega}(\vec{W}) - \delta_{\vec{W}}\ \vec{\omega}(\vec{U}) - \vec{\omega}([\vec{U},\vec{W}])$$

In our case:

$$d\vec{\omega}(\vec{U},\vec{W}) \equiv \delta_{\vec{U}}\delta_{\vec{W}}\vec{v} - \delta_{\vec{W}}\delta_{\vec{U}}\vec{v} - \delta_{[\vec{U},\vec{W}]}\vec{v}$$

So we are inclined to define the the Riemann curvature tensor as $\vec{R}(\vec{U},\vec{W})\vec{V} \equiv d\vec{\omega}(\vec{U},\vec{V})$, where $\vec{V}$ is embedded in the definition of $\vec{\omega}$.

We have a problem. Stokes theorem does not apply to vector valued forms in a curved space. This could block our strategy for analyzing curvature. Our goal was to express the change in a vector field around a loop. We can save our strategy by finding a scalar valued form that gives us enough information, like the rate of angular rotation of the vector along a path. This can be done. And, we can use Stokes theorem. And of course the expression for the Riemann curvature tensor appears in the exterior derivative of the scalar form.

Bottom line:
The Riemann curvature tensor is the exterior derivative of the covariant derivative of a vector field.